Introduction to Differential Geometry & General Relativity

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Lecture Notes
by
Stefan Waner

with a Special Guest Lecture
by Gregory C. Levine

Departments of Mathematics and Physics, Hofstra University

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These notes are dedicated to the memory of Hanno Rund.

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1. Preliminaries

Distance and Open Sets

Here, we do just enough topology so as to be able to talk about smooth manifolds. We begin with n-dimensional Euclidean space

$$E_n = \{(y_1, y_2, \ldots, y_n) \mid y_i \in \mathbb{R}\}.$$

Thus, E_1 is just the real line, E_2 is the Euclidean plane, and E_3 is 3−dimensional Euclidean space.

The **magnitude**, or **norm**, $\|\boldsymbol{y}\|$ of $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ in E_n is defined to be

$$\|\boldsymbol{y}\| = \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2}\,,$$

which we think of as its distance from the origin. Thus, the **distance** between two points $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$ in E_n is defined as the norm of $\boldsymbol{z} - \boldsymbol{y}$:

Distance Formula

Distance between $\boldsymbol{y}$ and $\boldsymbol{z} = \|\boldsymbol{z} - \boldsymbol{y}\| = \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2 + \ldots + (z_n - y_n)^2}$.

Proposition 1.1 (Properties of the norm)
The norm satisfies the following:
(a) $\|\boldsymbol{y}\| \geq 0$, and $\|\boldsymbol{y}\| = 0$ iff $\boldsymbol{y} = 0$ (positive definite)
(b) $\|\lambda\boldsymbol{y}\| = |\lambda|\|\boldsymbol{y}\|$ for every $\lambda \in \mathbb{R}$ and $\boldsymbol{y} \in E_n$.
(c) $\|\boldsymbol{y} + \boldsymbol{z}\| \leq \|\boldsymbol{y}\| + \|\boldsymbol{z}\|$ for every $\boldsymbol{y}, \boldsymbol{z} \in E_n$ (triangle inequality 1)
(d) $\|\boldsymbol{y} - \boldsymbol{z}\| \leq \|\boldsymbol{y} - \boldsymbol{w}\| + \|\boldsymbol{w} - \boldsymbol{z}\|$ for every $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in E_n$ (triangle inequality 2)

The proof of Proposition 1.1 is an exercise which may require reference to a linear algebra text (see "inner products").

Definition 1.2 A Subset U of E_n is called **open** if, for every $\boldsymbol{y}$ in U, all points of E_n within some positive distance r of $\boldsymbol{y}$ are also in U. (The size of r may depend on the point $\boldsymbol{y}$ chosen. Illustration in class).

Intuitively, an open set is a solid region minus its boundary. If we include the boundary, we get a **closed set**, which formally is defined as the complement of an open set.

Examples 1.3
(a) If $a \in E_n$, then the **open ball with center *a* and radius *r*** is the subset

$$B(\boldsymbol{a}, r) = \{x \in E_n \mid \|\boldsymbol{x} - \boldsymbol{a}\| < r\}.$$

Open balls are open sets: If $\boldsymbol{x} \in B(\boldsymbol{a}, r)$, then, with $s = r - \|\boldsymbol{x}-\boldsymbol{a}\|$, one has $B(\boldsymbol{x}, s) \subset B(\boldsymbol{a}, r)$.
(b) E_n is open.
(c) Ø is open.
(d) Unions of open sets are open.
(e) Open sets are unions of open balls. (Proof in class)

Definition 1.4 Now let $M \subset E_s$. A subset $V \subset M$ is called **open in *M*** (or **relatively open**) if, for every $\boldsymbol{y}$ in V, all points of M within some positive distance r of $\boldsymbol{y}$ are also in V.

Examples 1.5
(a) Open balls in *M*
If $M \subset E_s$, $\boldsymbol{m} \in M$, and $r > 0$, define

$$B_M(\boldsymbol{m}, r) = \{x \in M \mid \|\boldsymbol{x}-\boldsymbol{m}\| < r\}.$$

Then

$$B_M(\boldsymbol{m}, r) = B(\boldsymbol{m}, r) \cap M,$$

and so $B_M(\boldsymbol{m}, r)$ is open in M.
(b) M is open in M.
(c) Ø is open in M.
(d) Unions of open sets in M are open in M.
(e) Open sets in M are unions of open balls in M.

Parametric Paths and Surfaces in E_3

From now on, the three coordinates of 3-space will be referred to as y_1, y_2, and y_3.

Definition 1.6 A smooth **path** in E_3 is a set of three smooth (infinitely differentiable) real-valued functions of a single real variable t:

$$y_1 = y_1(t),\ y_2 = y_2(t),\ y_3 = y_3(t).$$

The variable t is called the **parameter** of the curve. The path is **non-singular** if the vector $\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}\right)$ is nowhere zero.

Notes
(a) Instead of writing $y_1 = y_1(t)$, $y_2 = y_2(t)$, $y_3 = y_3(t)$, we shall simply write $y_i = y_i(t)$.
(b) Since there is nothing special about three dimensions, we define a **smooth path in E_n** in exactly the same way: as a collection of smooth functions $y_i = y_i(t)$, where this time i goes from 1 to n.

Examples 1.7

(a) Straight lines in E_3

(b) Curves in E_3 (circles, etc.)

Definition 1.8 A **smooth surface embedded in E_3** is a collection of three smooth real-valued functions of *two* variables x^1 and x^2 (notice that x finally makes a debut).

$$y_1 = y_1(x^1, x^2)$$
$$y_2 = y_2(x^1, x^2)$$
$$y_3 = y_3(x^1, x^2),$$

or just

$$y_i = y_i(x^1, x^2) \quad (i = 1, 2, 3).$$

We also require that:

(a) The 3×2 matrix whose ij entry is $\dfrac{\partial y_i}{\partial x^j}$ has rank two.

(b) The associated function $E_2 \rightarrow E_3$ is a one-to-one map (that is, distinct points (x^1, x^2) in "parameter space" E_2 give different points (y_1, y_2, y_3) in E_3.

We call x^1 and x^2 the **parameters** or **local coordinates**.

Examples 1.9

(a) Planes in E_3

(b) The paraboloid $y_3 = y_1^2 + y_2^2$

(c) The sphere $y_1^2 + y_2^2 + y_3^2 = 1$, using spherical polar coordinates:

$$y_1 = \sin x^1 \cos x^2$$
$$y_2 = \sin x^1 \sin x^2$$
$$y_3 = \cos x^1$$

Note that condition (a) fails at $x^1 = 0$ and π.

(d) The ellipsoid $\dfrac{y_1^2}{a^2} + \dfrac{y_2^2}{b^2} + \dfrac{y_3^2}{c^2} = 1$, where a, b and c are positive constants.

(e) We calculate the rank of the Jacobean matrix for spherical polar coordinates.

(f) The torus with radii $a > b$:

$$y_1 = (a+b\cos x^2)\cos x^1$$
$$y_2 = (a+b\cos x^2)\sin x^1$$
$$y_3 = b\sin x^2$$

(Note that if $a \le b$ this torus is not embedded.)

(g) The functions

$$y_1 = x^1 + x^2$$
$$y_2 = x^1 + x^2$$
$$y_3 = x^1 + x^2$$

specify the *line* $y_1 = y_2 = y_3$ rather than a surface. Note that condition (a) fails here.

(h) The cone

$$y_1 = x^1$$
$$y_2 = x^2$$
$$y_3 = \sqrt{(x^1)^2 + (x^2)^2}$$

fails to be smooth at the origin (partial derivatives do not exist at the origin).

Question The parametric equations of a surface show us how to obtain a point on the surface once we know the two local coordinates (parameters). In other words, we have specified a function $E_2 \longrightarrow E_3$. How do we obtain the local coordinates from the Cartesian coordinates y_1, y_2, y_3?

Answer We need to solve for the local coordinates x^i as functions of y_j. This we do in one or two examples in class. For instance, in the case of a sphere, we get, for points other than (0, 0, +1):

$$x^1 = \cos^{-1}(y_3)$$
$$x^2 = \begin{cases} \cos^{-1}(y_1 / \sqrt{y_1^2+y_2^2}\,) & \text{if } y_2 \geq 0 \\ 2\pi - \cos^{-1}(y_1 / \sqrt{y_1^2+y_2^2}\,) & \text{if } y_2 < 0 \end{cases} .$$

(Note that x^2 is not defined at (0, 0, ±1).) This allows us to give each point on much of the sphere *two unique coordinates*, x^1, and x^2. There is a problem with continuity when $y_2 = 0$, since then x^2 switches from 0 to 2π. Thus, we restrict to the portion of the sphere given by

$0 < x^1 < \pi$	(North and South poles excluded)
$0 < x^2 < 2\pi$	(International Dateline excluded)

which is an open subset U of the sphere. (Think of it as the surface of the earth with the Greenwich Meridian removed.) We call x^1 and x^2 the **coordinate functions**. They are functions

$$x^1\colon U \longrightarrow E_1$$

and

$$x^2\colon U \longrightarrow E_1.$$

We can put them together to obtain a single function $\boldsymbol{x}\colon U \longrightarrow E_2$ given by

$$\boldsymbol{x}(y_1, y_2, y_3) = (x^1(y_1, y_2, y_3), x^2(y_1, y_2, y_3))$$
$$= \left(\cos^{-1}(y_3), \begin{cases} \cos^{-1}(y_1 / \sqrt{y_1^2+y_2^2}\,) & \text{if } y_2 \geq 0 \\ 2\pi - \cos^{-1}(y_1 / \sqrt{y_1^2+y_2^2}\,) & \text{if } y_2 < 0 \end{cases} \right)$$

as specified by the above formulas, as a **chart.**

Definition 1.10 A **chart** of a surface S is a pair of functions $\boldsymbol{x} = (x^1(y_1, y_2, y_3), x^2(y_1, y_2, y_3))$ which specify each of the **local coordinates** (parameters) x^1 and x^2 as smooth functions of a general point (**global** or **ambient coordinates**) (y_1, y_2, y_3) on the surface.

Question Why are these functions called a chart?
Answer The chart above assigns to each point on the sphere (away from the meridian) two coordinates. So, we can think of it as giving a two-dimensional map of the surface of the sphere, just like a geographic chart.

Question Our chart for the sphere is very nice, but is only appears to chart a portion of the sphere. What about the missing meridian?
Answer We can use another chart to get those by using different paramaterization that places the poles on the equator. (Diagram in class.)

In general, we chart an entire manifold M by "covering" it with open sets U which become the domains of coordinate charts.

Exercise Set 1

1. Prove Proposition 1.1.(Consult a linear algebra text.)
2. Prove the claim in Example 1.3 (d).
3. Prove that finite intersection of open sets in E_n are open.
4. Parametrize the following curves in E_3.
(a) a circle with center (1, 2, 3) and radius 4
(b) the curve $x = y^2$; $z = 3$
(c) the intersection of the planes $3x-3y+z=0$ and $4x+y+z=1$.
5. Express the following planes parametrically:

(a) $y_1 + y_2 - 2y_3 = 0$.
(b) $2y_1 + y_2 - y_3 = 12$.

6. Express the following quadratic surfaces parametrically: [Hint. For the hyperboloids, refer to parameterizations of the ellipsoid, and use the identity $\cosh^2 x - \sinh^2 x = 1$. For the double cone, use $y_3 = cx^1$, and x^1 as a factor of y_1 and y_2.]

(a) Hyperboloid of One Sheet: $\dfrac{y_1^2}{a^2} + \dfrac{y_2^2}{b^2} - \dfrac{y_3^2}{c^2} = 1$.

(b) Hyperboloid of Two Sheets: $\dfrac{y_1^2}{a^2} - \dfrac{y_2^2}{b^2} - \dfrac{y_3^2}{c^2} = 1$

(c) Cone: $\dfrac{y_3^2}{c^2} = \dfrac{y_1^2}{a^2} + \dfrac{y_2^2}{b^2}$.

(d) Hyperbolic Paraboloid: $\dfrac{y_3}{c} = \dfrac{y_1^2}{a^2} - \dfrac{y_2^2}{b^2}$

7. Solve the parametric equations you obtained in 5(a) and 6(b) for x^1 and x^2 as smooth functions of a general point (y_1, y_2, y_3) on the surface in question.

2. Smooth Manifolds and Scalar Fields

We now formalize the ideas in the last section.

Definition 2.1 An **open cover** of $M \subset E_s$ is a collection $\{U_\alpha\}$ of open sets in M such that $M = \cup_\alpha U_\alpha$.

Examples

(a) E_s can be covered by open balls.
(b) E_s can be covered by the single set E_s.
(c) The unit sphere in E_s can be covered by the collection $\{U_1, U_2\}$ where

$$U_1 = \{(y_1, y_2, y_3) \mid y_3 > -1/2\}$$
$$U_2 = \{(y_1, y_2, y_3) \mid y_3 < 1/2\}.$$

Definition 2.2 A subset M of E_s is called an ***n*-dimensional smooth manifold** if we are given a collection $\{U_\alpha; x_\alpha{}^1, x_\alpha{}^2, \ldots, x_\alpha{}^n\}$ where:

(a) The sets U_α form an open cover of M. U_α is called a **coordinate neighborhood** of M.

(b) Each $x_\alpha{}^r$ is a C^∞ real-valued function with domain U_α (that is, $x_\alpha{}^r\colon U_\alpha \longrightarrow E_1$).

(c) The map $x_\alpha\colon U_\alpha \longrightarrow E_n$ given by $x_\alpha(u) = (x_\alpha{}^1(u), x_\alpha{}^2(u), \ldots, x_\alpha{}^n(u))$ is one-to-one and has range an open set W_α in E_n.
x_α is called a **local chart of *M***, and $x_\alpha{}^r(u)$ is called the ***r*-th local coordinate** of the point u under the chart x_α.

(d) If (U, x^i), and $(V, \bar{x}^j)$ are two local charts of M, and if $U \cap V \neq \emptyset$, then noting that the one-to-one property allows us to express one set of parameters in terms of another:

$$x^i = x^i(\bar{x}^j)$$

with inverse

$$\bar{x}^k = \bar{x}^k(x^l),$$

we require these functions to be C^∞. These functions are called the **change-of-coordinates** functions.

The collection of all charts is called a **smooth atlas of *M***. The "big" space E_s in which the manifold M is embedded the **ambient space**.

Notes

1. Always think of the x^i as the **local coordinates** (or parameters) of the manifold. We can paramaterize each of the open sets U by using the inverse function x^{-1} of x, which assigns to each point in some open set of E_n a corresponding point in the manifold.

2. Condition **(c)** implies that

$$\det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0,$$

and

$$\det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \neq 0,$$

since the associated matrices must be invertible.

3. The ambient space need not be present in the general theory of manifolds; that is, it is possible to define a smooth manifold M without any reference to an ambient space at all—see any text on differential topology or differential geometry (or look at Rund's appendix).

4. More terminology: We shall sometimes refer to the x^i as the **local coordinates**, and to the y^j as the **ambient coordinates**. Thus, a point in an n-dimensional manifold M in E_s has n local coordinates, but s ambient coordinates.

5. We have put all the coordinate functions $x_\alpha{}^r$: $U_\alpha \longrightarrow E_1$ together to get a single map

$$x_\alpha: U_\alpha \longrightarrow W_\alpha \subset E_n.$$

A more elegant formulation of conditions (c) and (d) above is then the following: each W_α is an open subset of E_n, each x_α is invertible, and each composite

$$W_\alpha \xrightarrow{x_\alpha^{-1}} E_n \xrightarrow{x_\beta} W_\beta$$

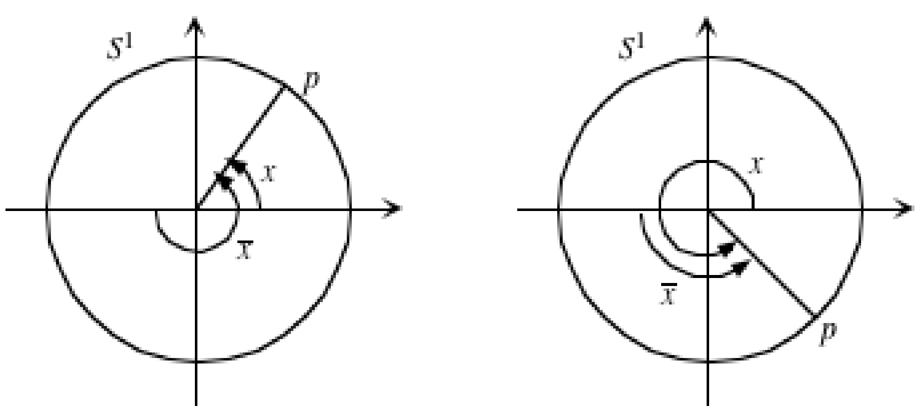
is smooth.

Examples 2.3

(a) E_n is an n-dimensional manifold, with the single identity chart defined by

$$x^i(y_1, \ldots, y_n) = y_i.$$

(b) S^1, the unit circle is a 1-dimensional manifold with charts given by taking the argument. Here is a possible structure:with two charts as show in in the following figure.



One has

$$x: S^1 - \{(1, 0)\} \longrightarrow E_1$$
$$\bar{x}: S^1 - \{(-1, 0)\} \longrightarrow E_1,$$

with $0 < x, \bar{x} < 2\pi$, and the change-of-coordinate maps are given by

$$\bar{x} = \begin{cases} x+\pi & \text{if } x < \pi \\ x-\pi & \text{if } x > \pi \end{cases}$$ (See the figure for the two cases.)

and

$$x = \begin{cases} \bar{x}+\pi & \text{if } \bar{x} < \pi \\ \bar{x}-\pi & \text{if } \bar{x} > \pi \end{cases} .$$

Notice the symmetry between x and $\bar{x}$. Also notice that these change-of-coordinate functions are only defined when $\theta \neq 0, \pi$. Further,

$$\partial\bar{x}/\partial x = \partial x/\partial\bar{x} = 1.$$

Note also that, in terms of complex numbers, we can write, for a point $p = e^{iz} \in S^1$,

$$x = \arg(z), \quad \bar{x} = \arg(-z).$$

(c) Generalized Polar Coordinates

Let us take $M = S^n$, the unit n−sphere,

$$S^n = \{(y_1, y_2, \dots , y_n, y_{n+1}) \in E_{n+1} \mid \Sigma_i y_i^2 = 1\},$$

with coordinates $(x^1, x^2, \dots , x^n)$ with

$$0 < x^1, x^2, \dots , x^{n-1} < \pi$$

and

$$0 < x^n < 2\pi,$$

given by

$$\begin{aligned}
y_1 &= \cos x^1 \\
y_2 &= \sin x^1 \cos x^2 \\
y_3 &= \sin x^1 \sin x^2 \cos x^3 \\
&\dots \\
y_{n-1} &= \sin x^1 \sin x^2 \sin x^3 \sin x^4 \dots \cos x^{n-1} \\
y_n &= \sin x^1 \sin x^2 \sin x^3 \sin x^4 \dots \sin x^{n-1} \cos x^n \\
y_{n+1} &= \sin x^1 \sin x^2 \sin x^3 \sin x^4 \dots \sin x^{n-1} \sin x^n
\end{aligned}$$

In the homework, you will be asked to obtain the associated chart by solving for the x^i. Note that if the sphere has radius r, then we can multiply all the above expressions by r, getting

$$\begin{aligned}
y_1 &= r \cos x^1 \\
y_2 &= r \sin x^1 \cos x^2
\end{aligned}$$

$y_3 = r \sin x^1 \sin x^2 \cos x^3$

$\ldots$

$y_{n-1} = r \sin x^1 \sin x^2 \sin x^3 \sin x^4 \ldots \cos x^{n-1}$

$y_n = r \sin x^1 \sin x^2 \sin x^3 \sin x^4 \ldots \sin x^{n-1} \cos x^n$

$y_{n+1} = r \sin x^1 \sin x^2 \sin x^3 \sin x^4 \ldots \sin x^{n-1} \sin x^n$.

(d) The torus $T = S^1 \times S^1$, with the following four charts:

$\boldsymbol{x}$: $(S^1-\{(1, 0)\})\times(S^1-\{(1, 0)\})\longrightarrow E_2$, given by

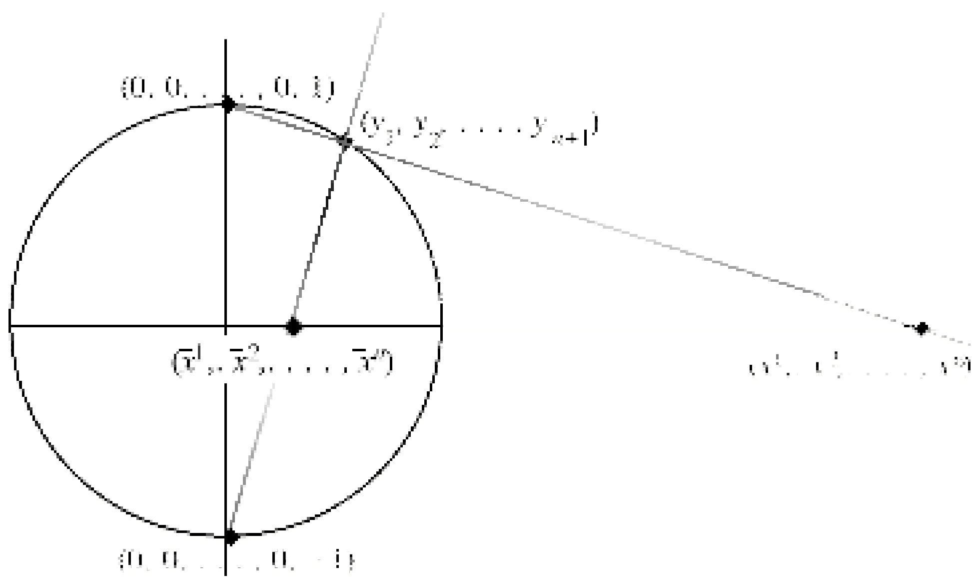
$x^1((\cos\theta, \sin\theta), (\cos\phi, \sin\phi)) = \theta$

$x^2((\cos\theta, \sin\theta), (\cos\phi, \sin\phi)) = \phi$.

The remaining charts are defined similarly, and the change-of-coordinate maps are omitted.

(e) The cylinder (homework)

(f) S^n, with (again) stereographic projection, is an n-manifold; the two charts are given as follows. Let P be the point $(0, 0, \ldots, 0, 1)$ and let Q be the point $(0, 0, \ldots, 0, -1)$. Then define two charts (S^n-P, x^i) and $(S^n-Q, \bar{x}^i)$ as follows. (See the figure.)



If $(y_1, y_2, \ldots, y_n, y_{n+1})$ is a point in S^n, let

$$x^1 = \frac{y_1}{1-y_{n+1}}; \qquad \bar{x}^1 = \frac{y_1}{1+y_{n+1}};$$
$$x^2 = \frac{y_2}{1-y_{n+1}}; \qquad \bar{x}^2 = \frac{y_2}{1+y_{n+1}};$$
$$\ldots \qquad \ldots$$
$$x^n = \frac{y_n}{1-y_{n+1}}. \qquad \bar{x}^n = \frac{y_n}{1+y_{n+1}}.$$

We can invert these maps as follows: Let $r^2 = \Sigma_i x^i x^i$, and $\bar{r}^2 = \Sigma_i \bar{x}^i \bar{x}^i$. Then:

$$y_1 = \frac{2x^1}{r^2+1}; \qquad y_1 = \frac{2\bar{x}^1}{1+\bar{r}^2};$$
$$y_2 = \frac{2x^2}{r^2+1}; \qquad y_2 = \frac{2\bar{x}^2}{1+\bar{r}^2};$$
$$\ldots \qquad \ldots$$
$$y_n = \frac{2x^n}{r^2+1}; \qquad y_n = \frac{2\bar{x}^n}{1+\bar{r}^2};$$
$$y_{n+1} = \frac{r^2-1}{r^2+1}; \qquad y_{n+1} = \frac{1-\bar{r}^2}{1+\bar{r}^2}.$$

The change-of-coordinate maps are therefore:

$$x^1 = \frac{y_1}{1-y_{n+1}} = \frac{\dfrac{2\bar{x}^1}{1+\bar{r}^2}}{1 - \dfrac{1-\bar{r}^2}{1+\bar{r}^2}} = \frac{\bar{x}^1}{\bar{r}^2}; \qquad \ldots\ldots (1)$$
$$x^2 = \frac{\bar{x}^2}{\bar{r}^2};$$
$$\ldots$$
$$x^n = \frac{\bar{x}^n}{\bar{r}^2}.$$

This makes sense, since the maps are not defined when $\bar{x}^i = 0$ for all i, corresponding to the north pole.

Note

Since $\bar{r}$ is the distance from $\bar{x}^i$ to the origin, this map is "hyperbolic reflection" in the unit circle: Equation (1) implies that x^i *and* $\bar{x}^i$ lie on the same ray from the origin, and

$$x^i = \frac{1}{\bar{r}} \frac{\bar{x}^i}{\bar{r}};$$

and squaring and adding gives

$$r = \frac{1}{\bar{r}}.$$

That is, project it to the circle, and invert the distance from the origin. This also gives the inverse relations, since we can write

$$\bar{x}^i = \bar{r}^2 x^i = \frac{x^i}{r^2}.$$

In other words, we have the following transformation rules.

Change of Coordinate Transformations for Stereographic Projection

Let $r^2 = \Sigma_i x^i x^i$, and $\bar{r}^2 = \Sigma_i \bar{x}^i \bar{x}^i$. Then

$$\bar{x}^i = \frac{x^i}{r^2}$$

$$x^i = \frac{\bar{x}^i}{\bar{r}^2}$$

$$r\bar{r} = 1$$

We now want to discuss scalar and vector fields on manifolds, but how do we specify such things? First, a scalar field.

Definition 2.4 A **smooth scalar field** on a smooth manifold M is just a smooth real-valued map Φ: $M \longrightarrow E_1$. (In other words, it is a smooth function of the coordinates of M as a subset of E_r.) Thus, Φ associates to each point m of M a unique scalar $\Phi(m)$. If U is a subset of M, then a **smooth scalar field on U** is smooth real-valued map Φ: $U \longrightarrow E_1$. If $U \neq M$, we sometimes call such a scalar field **local**.

If Φ is a scalar field on M and x is a chart, then we can express Φ as a smooth function ϕ of the associated parameters $x^1, x^2, \ldots, x^n$. If the chart is $\bar{x}$, we shall write $\bar{\phi}$ for the function of the other parameters $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$. Note that we must have $\phi = \bar{\phi}$ at each point of the manifold (see the "transformation rule" below).

Examples 2.5

(a) Let $M = E_n$ (with its usual structure) and let Φ be any smooth real-valued function in the usual sense. Then, using the identity chart, we have $\Phi = \phi$.

(b) Let $M = S^2$, and define $\Phi(y_1, y_2, y_3) = y_3$. Using stereographic projection, we find both ϕ and $\bar{\phi}$:

$$\phi(x^1, x^2) = y_3(x^1, x^2) = \frac{r^2-1}{r^2+1} = \frac{(x^1)^2 + (x^2)^2 - 1}{(x^1)^2 + (x^2)^2 + 1}$$

$$\bar{\phi}(\bar{x}^1, \bar{x}^2) = y_3(\bar{x}^1, \bar{x}^2) = \frac{1-\bar{r}^2}{1+\bar{r}^2} = \frac{1 - (\bar{x}^1)^2 - (\bar{x}^2)^2}{1 + (\bar{x}^1)^2 + (\bar{x}^2)^2}$$

(c) Local Scalar Field The most obvious candidate for local fields are the coordinate functions themselves. If U is a coordinate neighborhood, and $\boldsymbol{x} = \{x^i\}$ is a chart on U, then the maps x^i are local scalar fields.

Sometimes, as in the above example, we may wish to specify a scalar field purely by specifying it in terms of its local parameters; that is, by specifying the various functions ϕ instead of the single function Φ. The problem is, we can't just specify it any way we want, since it must give a value to each point in the manifold independently of local coordinates. That is, if a point $p \in M$ has local coordinates (x^j) with one chart and $(\bar{x}^h)$ with another, they must be related via the relationship

$$\bar{x}^j = \bar{x}^j(x^h).$$

Transformation Rule for Scalar Fields

$$\bar{\phi}(\bar{x}^j) = \phi(x^h)$$

whenever (x^h) and $(\bar{x}^j)$ are the coordinates under x and $\bar{x}$ of some point p in M. This formula can also be read as

$$\bar{\phi}(\bar{x}^j(x^h)) = \phi(x^h)$$

Example 2.6 Look at Example 2.5(b) above. If you substituted $\bar{x}^i$ as a function of the x^j, you would get $\bar{\phi}(\bar{x}^1, \bar{x}^2) = \phi(x^1, x^2)$.

Exercise Set 2

1. Give the paraboloid $z = x^2 + y^2$ the structure of a smooth manifold.

2. Find a smooth atlas of E_2 consisting of three charts.

3. **(a)** Extend the method in Exercise 1 to show that the graph of any smooth function f: $E_2 \longrightarrow E_1$ can be given the structure of a smooth manifold.

(b) Generalize part (a) to the graph of a smooth function f: $E_n \longrightarrow E_1$.

4. Two atlases of the manifold M **give the same smooth structure** if their union is again a smooth atlas of M.

(a) Show that the smooth atlases (E_1, f), and (E_1, g), where $f(x) = x$ and $g(x) = x^3$ are incompatible.

(b) Find a third smooth atlas of E_1 that is incompatible with both the atlases in part (a).

5. Consider the ellipsoid $L \subset E_3$ specified by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a, b, c \neq 0).$$

Define f: $L \longrightarrow S^2$ by $f(x, y, z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$.

(a) Verify that f is invertible (by finding its inverse).

(b) Use the map f, together with a smooth atlas of S^2, to construct a smooth atlas of L.

6. Find the chart associated with the generalized spherical polar coordinates described in Example 2.3(c) by inverting the coordinates. How many additional charts are needed to get an atlas? Give an example.

7. Obtain the equations in Example 2.3(f).

3. Tangent Vectors and the Tangent Space

We now turn to vectors tangent to smooth manifolds. We must first talk about smooth paths on M.

Definition 3.1 A **smooth path** on M is a smooth map $\boldsymbol{r}$: $J \to M$, where J is some open interval. (Thus, $\boldsymbol{r}(t) = (y_1(t), y_2(t), \ldots, y_s(t)$ for $t \in J$.) We say that r is a smooth path **through $m \in M$** if $\boldsymbol{r}(t_0) = m$ for some $t_0 \in J$. We can specify a path in M at m by its coordinates:

$$
\begin{aligned}
y_1 &= y_1(t), \\
y_2 &= y_2(t), \\
&\ldots \\
y_s &= y_s(t),
\end{aligned}
$$

where m is the point $(y_1(t_0), y_2(t_0), \ldots, y_s(t_0))$. Equivalently, since the ambient and local coordinates are functions of each other, we can also express a path—at least that part of it inside a coordinate neighborhood—in terms of its local coordinates:

$$
\begin{aligned}
x^1 &= x^1(t), \\
x^2 &= x^2(t), \\
&\ldots \\
x^n &= x^n(t).
\end{aligned}
$$

Examples 3.2
(a) Smooth paths in E_n
(b) A smooth path in S^1, and S^n

Definition 3.3 A **tangent vector** at $m \in M \subset E_r$ is a vector $\boldsymbol{v}$ in E_r of the form

$$\boldsymbol{v} = \boldsymbol{y}'(t_0)$$

for some path $\boldsymbol{y} = \boldsymbol{y}(t)$ in M through m and $\boldsymbol{y}(t_0) = m$.

Examples 3.4
(a) Let M be the surface $y_3 = y_1^2 + y_2^2$, which we parameterize by

$$
\begin{aligned}
y_1 &= x^1 \\
y_2 &= x^2 \\
y_3 &= (x^1)^2 + (x^2)^2
\end{aligned}
$$

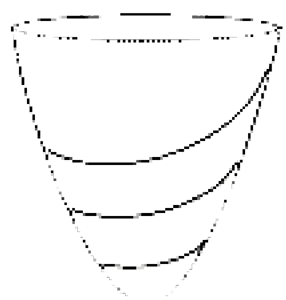
This corresponds to the single chart $(U{=}M; x^1, x^2)$, where

$x^1 = y_1$ and $x^2 = y_2$.

To specify a tangent vector, let us first specify a path in M, such as, for $t \in (0, +\infty)$

$$y_1 = \sqrt{t}\, \sin t$$
$$y_2 = \sqrt{t}\, \cos t$$
$$y_3 = t$$

(Check that the equation of the surface is satisfied.) This gives the path shown in the figure.



Now we obtain a tangent vector field along the path by taking the derivative:

$$\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}\right) = \left(\sqrt{t}\, \cos t + \frac{\sin t}{2\sqrt{t}},\ -\sqrt{t}\, \sin t + \frac{\cos t}{2\sqrt{t}},\ 1\right).$$

(To get actual tangent vectors at points in M, evaluate this at a fixed point t_0.)

Note We can also express the coordinates x^i in terms of t:

$$x^1 = y_1 = \sqrt{t}\, \sin t$$
$$x^2 = y_2 = \sqrt{t}\, \cos t$$

This descibes a path in some chart (that is, in coordinate space E_n) rather than on the mnanifold itself. We can also take the derivative,

$$\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = \left(\sqrt{t}\, \cos t + \frac{\sin t}{2\sqrt{t}},\ -\sqrt{t}\, \sin t + \frac{\cos t}{2\sqrt{t}}\right).$$

We also think of this as the tangent vector, given in terms of the local coordinates. A lot more will be said about the relationship between the above two forms of the tangent vector below.

Algebra of Tangent Vectors: Addition and Scalar Multiplication

The sum of two tangent vectors is, geometrically, also a tangent vector, and the same goes for scalar multiples of tangent vectors. However, we have defined tangent vectors using paths in M, and we cannot produce these new vectors by simply adding or scalar-multiplying the corresponding paths: if $\mathbf{y} = \mathbf{f}(t)$ and $\mathbf{y} = \mathbf{g}(t)$ are two paths through $m \in M$ where $\mathbf{f}(t_0) = \mathbf{g}(t_0) = m$, then adding them coordinate-wise need not produce a path in M. However, we *can* add these paths using some chart as follows.

Choose a chart x at m, with the property (for convenience) that $x(m) = \mathbf{0}$. Then the paths $x(\mathbf{f}(t))$ and $x(\mathbf{g}(t))$ (defined as in the note above) give two paths through the origin in coordinate space. *Now* we can add these paths or multiply them by a scalar without leaving coordinate space and then use the chart map to lift the result back up to M. In other words, define

$$(\mathbf{f}+\mathbf{g})(t) = x^{-1}(x(\mathbf{f}(t)) + x(\mathbf{g}(t))$$

and $$(\lambda \mathbf{f})(t) = x^{-1}(\lambda x(\mathbf{f}(t))).$$

Taking their derivatives at the point t_0 will, by the chain rule, produce the sum and scalar multiples of the corresponding tangent vectors.

Definition 3.5 If M is an n-dimensional manifold, and $m \in M$, then the **tangent space at *m*** is the set T_m of all tangent vectors at m.

Since we have equipped T_m with addition and scalar multiplication satisfying the "usual" properties, T_m has the structure of a **vector space**.

Let us return to the issue of the two ways of describing the coordinates of a tangent vector at a point $m \in M$: writing the path as $y_i = y_i(t)$ we get the **ambient coordinates** of the tangent vector:

$$\mathbf{y}'(t_0) = \left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \ldots\ldots, \frac{dy_s}{dt}\right)_{t=t_0} \qquad \text{Ambient coordinates}$$

and, using some chart x at m, we get the **local coordinates**

$$\mathbf{x}'(t_0) = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}\right)_{t=t_0}. \qquad \text{Local Coordinates}$$

Question In general, how are the dx^i/dt related to the dy_i/dt?

Answer By the chain rule,

$$\frac{dy_1}{dt} = \frac{\partial y_1}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial y_1}{\partial x^2}\frac{dx^2}{dt} + \ldots + \frac{\partial y_1}{\partial x^n}\frac{dx^n}{dt}$$

and similarly for dy_2/dt ... dy_n/dt. Thus, *we can recover the original ambient vector coordinates from the local coordinates*. In other words, *the local vector coordinates completely specify the tangent vector*.

Note We use this formula to convert local coordinates to ambient coordinates:

Converting Between Local and Ambient Coordinates of a Tangent Vector

If the tangent vector V has ambient coordinates $(v_1, v_2, \ldots, v_s)$ and local coordinates $(v^1, v^2, \ldots, v^n)$, then they are related by the formulæ

$$v_i = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x^k} v^k$$

and

$$v^i = \sum_{k=1}^{s} \frac{\partial x^i}{\partial y_k} v_k$$

Note To obtain the coordinates of sums or scalar multiples of tangent vectors, simply take the corresponding sums and scalar multiples of the coordinates. In other words:

$$(v+w)^i = v^i + w^i$$

and $$(\lambda v)^i = \lambda v^I$$

just as we would expect to do for ambient coordinates. (Why can we do this?)

Examples 3.4 Continued:

(b) Take $M = E_n$, and let $\boldsymbol{v}$ be any vector in the usual sense with coordinates α^i. Choose x to be the usual chart $x^i = y_i$. If $\boldsymbol{p} = (p^1, p^2, \ldots, p^n)$ is a point in M, then $\boldsymbol{v}$ is the derivative of the path

$$\begin{aligned} x^1 &= p^1 + t\alpha^1 \\ x^2 &= p^2 + t\alpha^2; \\ &\ldots \\ x^n &= p^n + t\alpha^n \end{aligned}$$

at $t = 0$. Thus this vector has local and ambient coordinates equal to each other, and equal to

$$\frac{dx^i}{dt} = \alpha^i,$$

which are the same as the original coordinates. In other words, the tangent vectors are "the same" as ordinary vectors in E_n.

(c) Let $M = S^2$, and the path in S^2 given by

$$\begin{aligned} y_1 &= \sin t \\ y_2 &= 0 \\ y_3 &= \cos t \end{aligned}$$

This is a path (circle) through $m = (0, 0, 1)$ following the line of longitude $\phi = x^2 = 0$, and has tangent vector

$$\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}\right) = (\cos t,\ 0,\ -\sin t) = (1, 0, 0) \text{ at the point } m.$$

(d) We can also use the local coordinates to describe a path; for instance, the path in part (c) can be described using spherical polar coordinates by

$$\begin{aligned} x^1 &= t \\ x^2 &= 0 \end{aligned}$$

The derivative

$$\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = (1, 0)$$

gives the local coordinates of the tangent vector itself (the coordinates of its image in coordinate Euclidean space).

(e) In general, if $(U;\ x^1, x^2, \ldots, x^n)$ is a coordinate system near m, then we can obtain paths $y_i(t)$ by setting

$$x^j(t) = \begin{cases} t + \text{const.} & \text{if } j = i \\ \text{const.} & \text{if } j \neq i \end{cases},$$

where the constants are chosen to make $x^i(t_0)$ correspond to m for some t_0. (The paths in (c) and (d) are an example of this.) To view this as a path in M, we just apply the parametric equations $y_i = y_i(x^j)$, giving the y_i as functions of t.

The associated tangent vector at the point where $t = t_0$ is called $\partial/\partial x^i$. It has local coordinates

$$v^j = \left(\frac{dx^j}{dt}\right)_{t = t_0} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} = \delta_i^j$$

δ_i^j is called the **Kronecker Delta**, and is defined by

$$\delta_i^j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad .$$

We can now get the ambient coordinates by the above conversion:

$$v_j = \sum_{k=1}^{n} \frac{\partial y_j}{\partial x^k} v^k = \sum_{k=1}^{n} \frac{\partial y_j}{\partial x^k} \delta_i^k = \frac{\partial y_j}{\partial x^i} .$$

We call this vector $\frac{\partial}{\partial x^i}$. Summarizing,

Definition of $\frac{\partial}{\partial x^i}$

Pick a point $m \in M$. Then $\frac{\partial}{\partial x^i}$ is the vector at m whose *local coordinates* under x are given by

$$j \text{ th coordinate} = \left(\frac{\partial}{\partial x^i}\right)^j$$

$$= \delta_i^j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad \text{(Local coords of } \partial/\partial x^i\text{)}$$

$$= \frac{\partial x^j}{\partial x^i}$$

Its *ambient coordinates* are given by

$$j \text{ th coordinate} = \frac{\partial y_j}{\partial x^i} \qquad \text{(Ambient coords of } \partial/\partial x^i\text{)}$$

(everything evaluated at t_0) Notice that the path itself has disappeared from the definition...