become a common view. That something of the same kind[2] may be going on under our feet is a supposition not wholly lacking evidential support. The evolution of considerable percentages[3] of helium in natural gas will be recalled in this connection. The observed temperature gradient from the surface down can hardly continue very far (as it must if, for example, it were due to radioactive materials uniformly distributed); but it might very well take its origin in a zone of subatomic activity at a moderate distance from the face of the earth.

* Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 86.

[1] *J. Amer. Chem. Soc.*, 1917, p. 867.

[2] Perhaps a balance or equilibrium between integration and disintegration, under the influence of temperature and pressure, and subject to the principle of Le Chatelier.

[3] As much as 2 per cent has been recorded, a quantity that seems too great to be attributable to radioactive transformations. In connection with this and other points related to the subject here discussed, compare the following: V. Moritz, Der Stoffwechsel der Erde, *Zs. Elektrochemie*, **8**, 1922, pp. 411–421; S. Arrhenius, *Ibid.*, pp. 404–411; W. Vernadsky, *Nature*, August 12, 1922, p. 229. F. W. Clarke, The Data of Geochemistry, *U. S. Geological Survey Bulletin*, No. 695, p. 35; H. S. Washington, *J. Franklin Inst.*, **190**, 1920, pp. 757–815; Sir E. Rutherford, Thirteenth Kelvin Lecture, Electricity and Matter (Published in *Nature* Aug. 5, 1922, p. 182); R. B. Moore, *Nature*, Jan. 20, 1923, pp. 88, 91. It should perhaps be noted here that the table of the abundance of the elements given by Washington loc. cit., p. 777 does not include the hydrosphere and the atmosphere.

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# *CONTINUOUS TRANSFORMATIONS OF MANIFOLDS*

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1. For about a decade continuous transformations of surfaces and manifolds have been investigated by various authors (Brouwer, Birkhoff, and others). Very recently Alexander and Birkhoff-Kellogg published on the question two interesting papers with some notable applications.[1]

The problems that present themselves are mainly two: (*a*) Determination of the minimum number of fixed points and related questions. (*b*) Classification into classes of continuous transformations.

We propose to give a new and far reaching method for the attack of these two problems (No. 3). Only brief indications will be found here, a complete discussion being reserved for a later occasion.

2. Let $W_n$ be an $n$ dimensional manifold such as considered by Veblen in his Cambridge Colloquium lectures on Analysis Situs, Ch. 3. More specifically we assume that it is closed, homogeneous, two sided. Let

$M_k$, $M_{n-k}$ be two other two sided manifolds contained in $W_n$ and of dimensionality equal to the subscript. By means of indicatrices we may attach a sign to each of their points of intersection. (Poincaré and Kronecker.) Let there be $q$ points affected with $+$, $q'$ with $-$. We shall call $q - q'$ the *algebraic number of intersections* and denote it by $(M_k M_{n-k})$ as against the *arithmetic* number $q + q'$. It is assumed of course that $q$ and $q'$ are finite and that all points of intersection are *non singular* for the manifolds, which moreover have no *contacts*.

Let $\Gamma_k$, $\Gamma_{n-k}$, be, respectively, a $k$-cycle and an $(n-k)$-cycle of $W_n$. If necessary they shall be slightly deformed in order to insure that they intersect as indicated above. We then have these two properties:

(*a*) If $\Gamma_k$ or $\Gamma_{n-k} \sim 0$, mod. $W_n$, then $(\Gamma_k \Gamma_{n-k}) = 0$.

(*b*) If $\Gamma_k \sim \Gamma'_k + \Gamma''_k$, mod. $W_n$, then $(\Gamma_k \Gamma_{n-k}) = (\Gamma'_k \Gamma_{n-k}) + (\Gamma''_k \Gamma_{n-k})$.

Similarly for $\Gamma_{n-k}$.

Let now $\Gamma^i_k$, $(i = 1, 2, \ldots)$, $\Gamma^j_{n-k}$ $(j = 1, 2, \ldots)$ be two fundamental systems for our cycles. We shall have

$$\Gamma_k \sim \Sigma x_i \Gamma^i_k, \; \Gamma_{n-k} \sim \Sigma y_j \Gamma^j_{n-k}; \qquad \therefore \; (\Gamma_k \Gamma_{n-k}) = \Sigma x_i y_j (\Gamma^i_k \Gamma^j_{n-k}).$$

It follows that in order to know the algebraic number of intersections we need: (*a*) their expressions in terms of their respective fundamental systems; (*b*) the algebraic numbers of intersections of any two cycles of the fundamental systems.

3. We are now prepared to consider the problem of continuous transformations.—Let $W'_n$ be another copy of $W_n$ and $W_{2n}$ the image of all point pairs of $W_n$ and $W'_n$. Let $T$ be a continuous univalued transformation of $W_n$ into itself (its inverse need not be univalued), $A$ a point of $W_n$, $B'$ the image of $B = T.A$ on $W_n'$, $C$ that of the point pair $A$, $B'$, on $W_{2n}$. *The locus of $C$ is an $n$-cycle $\Gamma_n$ of $W_{2n}$, complete image of $T$*, in the sense that its points are in one to one correspondence with the point pairs $A$, $B = T.A$.

If $T$ varies in a continuous sustem, $\Gamma_n$ will vary in a continuous system of homologous cycles. Hence *a class of transformations is characterized by a class of homologous cycles* of $W_{2n}$. Of course we do not affirm that to every $\Gamma_n$ corresponds a suitable $T$, but it is not difficult to describe the cycles for which there exists a $T$. Suffice to state that *every $\Gamma$ corresponding to a $T$ may be approximated as nearly as we please by a cycle sum of analytical parts which also belongs to some $T$*. It is readily seen that in the sequel $\Gamma_n$ may be replaced by the new cycle to which the discussion of the No. 2 applies in full, and this will be assumed once for all.

Let $T'$ be another transformation, $\Gamma'_n$ the corresponding cycle. The number of points $A$ whose transform by $T$ or $T'$ are the same, is equal to the *arithmetic* number of intersections of $\Gamma_n$ and $\Gamma'_n$. But this number