Mathematical Background: Foundations of Infinitesimal Calculus

second edition

by

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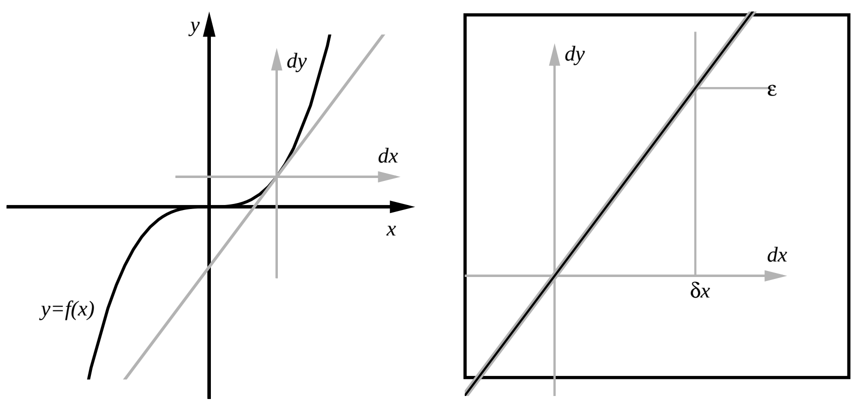

Figure 0.1: A Microscopic View of the Tangent

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Preface to the Mathematical Background

We want you to reason with mathematics. We are not trying to get everyone to give formalized proofs in the sense of contemporary mathematics; ‘proof’ in this course means ‘convincing argument.’ We expect you to use correct reasoning and to give careful explanations. The projects bring out these issues in the way we find best for most students, but the pure mathematical questions also interest some students. This book of mathematical “background” shows how to fill in the mathematical details of the main topics from the course. These proofs are completely rigorous in the sense of modern mathematics – technically bulletproof. We wrote this book of foundations in part to provide a convenient reference for a student who might like to see the “theorem - proof” approach to calculus.

We also wrote it for the interested instructor. In re-thinking the presentation of beginning calculus, we found that a simpler basis for the theory was both possible and desirable. The pointwise approach most books give to the theory of derivatives spoils the subject. Clear simple arguments like the proof of the Fundamental Theorem at the start of Chapter 5 below are not possible in that approach. The result of the pointwise approach is that instructors feel they have to either be dishonest with students or disclaim good intuitive approximations. This is sad because it makes a clear subject seem obscure. It is also unnecessary – by and large, the intuitive ideas work provided your notion of derivative is strong enough. This book shows how to bridge the gap between intuition and technical rigor.

A function with a positive derivative ought to be increasing. After all, the slope is positive and the graph is supposed to look like an increasing straight line. How could the function NOT be increasing? Pointwise derivatives make this bizarre thing possible - a positive “derivative” of a non-increasing function. Our conclusion is simple. That definition is WRONG in the sense that it does NOT support the intended idea.

You might agree that the counterintuitive consequences of pointwise derivatives are unfortunate, but are concerned that the traditional approach is more “general.” Part of the point of this book is to show students and instructors that nothing of interest is lost and a great deal is gained in the straightforward nature of the proofs based on “uniform” derivatives. It actually is not possible to give a *formula* that is pointwise differentiable and not uniformly differentiable. The pieced together pointwise counterexamples seem contrived and out-of-place in a course where students are learning valuable new rules. It is a theorem that derivatives computed by rules are automatically continuous where defined. We want the course development to emphasize good intuition and positive results. This background shows that the approach is sound.

This book also shows how the pathologies arise in the traditional approach – we left pointwise pathology out of the main text, but present it here for the curious and for comparison. Perhaps only math majors ever need to know about these sorts of examples, but they are fun in a negative sort of way.

This book also has several theoretical topics that are hard to find in the literature. It includes a complete self-contained treatment of Robinson’s modern theory of infinitesimals, first discovered in 1961. Our simple treatment is due to H. Jerome Keisler from the 1970’s. Keisler’s elementary calculus using infinitesimals is sadly out of print. It used pointwise derivatives, but had many novel ideas, including the first modern use of a microscope to describe the derivative. (The l’Hospital/Bernoulli calculus text of 1696 said curves consist of infinitesimal straight segments, but I do not know if that was associated with a magnifying transformation.) Infinitesimals give us a very simple way to understand the uniform derivatives, although this can also be clearly understood using function limits as in the text by Lax, et al, from the 1970s. Modern graphical computing can also help us "see" graphs converge as stressed in our main materials and in the interesting Uhl, Porta, Davis, *Calculus & Mathematica* text.

Almost all the theorems in this book are well-known old results of a carefully studied subject. The well-known ones are more important than the few novel aspects of the book. However, some details like the converse of Taylor's theorem – both continuous and discrete – are not so easy to find in traditional calculus sources. The microscope theorem for differential equations does not appear in the literature as far as we know, though it is similar to research work of Francine and Marc Diener from the 1980s.

We conclude the book with convergence results for Fourier series. While there is nothing novel in our approach, these results have been lost from contemporary calculus and deserve to be part of it. Our development follows Courant's calculus of the 1930s giving wonderful results of Dirichlet's era in the 1830s that clearly settle some of the convergence mysteries of Euler from the 1730s. This theory and our development throughout is usually easy to apply. "Clean" theory should be the servant of intuition – building on it and making it stronger and clearer.

There is more that is novel about this "book." It is free and it is not a "book" since it is not printed. Thanks to small marginal cost, our publisher agreed to include this electronic text on CD at no extra cost. We also plan to distribute it over the world wide web. We hope our fresh look at the foundations of calculus will stimulate your interest. Decide for yourself what's the best way to understand this wonderful subject. Give your own proofs.

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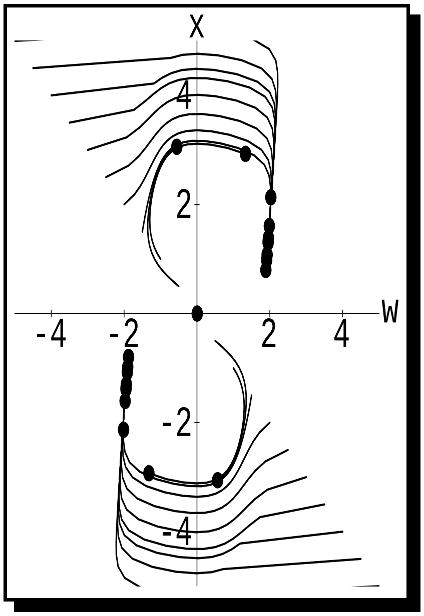

Part 1

Numbers and Functions

CHAPTER 1

Numbers

This chapter gives the algebraic laws of the number systems used in calculus.

Numbers represent various idealized measurements. Positive integers may count items, fractions may represent a part of an item or a distance that is part of a fixed unit. Distance measurements go beyond rational numbers as soon as we consider the hypotenuse of a right triangle or the circumference of a circle. This extension is already in the realm of imagined "perfect" measurements because it corresponds to a perfectly straight-sided triangle with perfect right angle, or a perfectly round circle. Actual real measurements are always rational and have some error or uncertainty.

The various "imaginary" aspects of numbers are very useful fictions. The rules of computation with perfect numbers are much simpler than with the error-containing real measurements. This simplicity makes fundamental ideas clearer.

Hyperreal numbers have 'teeny tiny numbers' that will simplify approximation estimates. Direct computations with the ideal numbers produce symbolic approximations equivalent to the function limits needed in differentiation theory (that the rules of Theorem 1.12 give a direct way to compute.) Limit theory does not give the answer, but only a way to justify it once you have found it.

1.1 Field Axioms

The laws of algebra follow from the field axioms. This means that algebra is the same with Dedekind's "real" numbers, the complex numbers, and Robinson's "hyperreal" numbers.

Axiom 1.1. *Field Axioms*

A "field" of numbers is any set of objects together with two operations, addition and multiplication where the operations satisfy:

- *The commutative laws of addition and multiplication,*

$$a_1 + a_2 = a_2 + a_1 \qquad \& \qquad a_1 \cdot a_2 = a_2 \cdot a_1$$

- *The associative laws of addition and multiplication,*

$$a_1 + (a_2 + a_3) = (a_1 + a_2) + a_3 \qquad \& \qquad a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$$

- *The distributive law of multiplication over addition,*

$$a_1 \cdot (a_2 + a_3) = a_1 \cdot a_2 + a_1 \cdot a_3$$

- *There is an additive identity, 0, with $0 + a = a$ for every number a.*
- *There is an multiplicative identity, 1, with $1 \cdot a = a$ for every number $a \neq 0$.*
- *Each number a has an additive inverse, $-a$, with $a + (-a) = 0$.*
- *Each nonzero number a has a multiplicative inverse, $\frac{1}{a}$, with $a \cdot \frac{1}{a} = 1$.*

A computation needed in calculus is

Example 1.1. *The Cube of a Binomial*

$$\begin{aligned}(x + \Delta x)^3 &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 \\ &= x^3 + 3x^2\Delta x + (\Delta x(3x + \Delta x))\,\Delta x\end{aligned}$$

We analyze the term $\varepsilon = (\Delta x(3x + \Delta x))$ in differentiation.

The reader could laboriously demonstrate that only the field axioms are needed to perform the computation. This means it holds for rational, real, complex, or hyperreal numbers. Here is a start. Associativity is needed so that the cube is well defined, or does not depend on the order we multiply. We use this in the next computation, then use the distributive property, the commutativity and the distributive property again, and so on.

$$\begin{aligned}(x + \Delta x)^3 &= (x + \Delta x)(x + \Delta x)(x + \Delta x) \\ &= (x + \Delta x)((x + \Delta x)(x + \Delta x)) \\ &= (x + \Delta x)((x + \Delta x)x + (x + \Delta x)\Delta x) \\ &= (x + \Delta x)((x^2 + x\Delta x) + (x\Delta x + \Delta x^2)) \\ &= (x + \Delta x)(x^2 + x\Delta x + x\Delta x + \Delta x^2) \\ &= (x + \Delta x)(x^2 + 2x\Delta x + \Delta x^2) \\ &= (x + \Delta x)x^2 + (x + \Delta x)2x\Delta x + (x + \Delta x)\Delta x^2) \\ &\ \vdots\end{aligned}$$

The natural counting numbers $1, 2, 3, \ldots$ have operations of addition and multiplication, but do not satisfy all the properties needed to be a field. Addition and multiplication do satisfy the commutative, associative, and distributive laws, but there is no additive inverse

0 in the counting numbers. In ancient times, it was controversial to add this element that could stand for counting nothing, but it is a useful fiction in many kinds of computations.

The negative integers $-1, -2, -3, \ldots$ are another idealization added to the natural numbers that make additive inverses possible - they are just new numbers with the needed property. Negative integers have perfectly concrete interpretations such as measurements to the left, rather than the right, or amounts owed rather than earned.

The set of all integers; positive, negative, and zero, still do not form a field because there are no multiplicative inverses. Fractions, $\pm 1/2$, $\pm 1/3$, ... are the needed additional inverses. When they are combined with the integers through addition, we have the set of all rational numbers of the form $\pm p/q$ for natural numbers p and $q \neq 0$. The rational numbers are a field, that is, they satisfy all the axioms above. In ancient times, rationals were sometimes considered only "operators" on "actual" numbers like $1, 2, 3, \ldots$.

The point of the previous paragraphs is simply that we often extend one kind of number system in order to have a new system with useful properties. The complex numbers extend the field axioms above beyond the "real" numbers by adding a number $\mathbf{i}$ that solves the equation $x^2 = -1$. (See the CD Chapter 29 of the main text.) Hundreds of years ago this number was controversial and is still called "imaginary." In fact, all numbers are useful constructs of our imagination and some aspects of Dedekind's "real" numbers are much more abstract than $\mathbf{i}^2 = -1$. (For example, since the reals are "uncountable," "most" real numbers have no description what-so-ever.)

The rationals are not "complete" in the sense that the linear measurement of the side of an equilateral right triangle ($\sqrt{2}$) cannot be expressed as p/q for p and q integers. In Section 1.3 we "complete" the rationals to form Dedekind's "real" numbers. These numbers correspond to perfect measurements along an ideal line with no gaps.

The complex numbers cannot be ordered with a notion of "smaller than" that is compatible with the field operations. Adding an "ideal" number to serve as the square root of -1 is not compatible with the square of every number being positive. When we make extensions beyond the real number system we need to make choices of the kind of extension depending on the properties we want to preserve.

Hyperreal numbers allow us to compute estimates or limits directly, rather than making inverse proofs with inequalities. Like the complex extension, hyperreal extension of the reals loses a property; in this case completeness. Hyperreal numbers are explained beginning in Section 1.4 below and then are used extensively in this background book to show how many intuitive estimates lead to simple direct proofs of important ideas in calculus.

The hyperreal numbers (discovered by Abraham Robinson in 1961) are still controversial because they contain infinitesimals. However, they are just another extended modern number system with a desirable new property. Hyperreal numbers can help you understand limits of real numbers and many aspects of calculus. Results of calculus could be proved without infinitesimals, just as they could be proved without real numbers by using only rationals. Many professors still prefer the former, but few prefer the latter. We believe that is only because Dedekind's "real" numbers are more familiar than Robinson's, but we will make it clear how both approaches work as a theoretical background for calculus.

There is no controversy concerning the logical soundness of hyperreal numbers. The use of infinitesimals in the early development of calculus beginning with Leibniz, continuing with Euler, and persisting to the time of Gauss was problematic. The founders knew that their use of infinitesimals was logically incomplete and could lead to incorrect results. Hyperreal numbers are a correct treatment of infinitesimals that took nearly 300 years to discover.

With hindsight, they also have a simple description. The Function Extension Axiom 2.1 explained in detail in Chapter 2 was the missing key.

Exercise set 1.1

1. *Show that the identity numbers* 0 *and* 1 *are unique. (HINT: Suppose* $0' + a = a$*. Add* $-a$ *to both sides.)*
2. *Show that* $0 \cdot a = 0$*. (HINT: Expand* $\left(0 + \frac{b}{a}\right) \cdot a$ *with the distributive law and show that* $0 \cdot a + b = b$*. Then use the previous exercise.)*
3. *The inverses* $-a$ *and* $\frac{1}{a}$ *are unique. (HINT: Suppose not,* $0 = a - a = a + b$*. Add* $-a$ *to both sides and use the associative property.)*
4. *Show that* $-1 \cdot a = -a$*. (HINT: Use the distributive property on* $0 = (1-1) \cdot a$ *and use the uniqueness of the inverse.)*
5. *Show that* $(-1) \cdot (-1) = 1$*.*
6. *Other familiar properties of algebra follow from the axioms, for example, if* $a_3 \neq 0$ *and* $a_4 \neq 0$*, then*

$$\frac{a_1 + a_2}{a_3} = \frac{a_1}{a_3} + \frac{a_2}{a_3} \qquad , \qquad \frac{a_1 \cdot a_2}{a_3 \cdot a_4} = \frac{a_1}{a_3} \cdot \frac{a_2}{a_4} \qquad \& \qquad a_3 \cdot a_4 \neq 0$$

1.2 Order Axioms

Estimation is based on the inequality $\leq$ *of the real numbers.*

One important representation of rational and real numbers is as measurements of distance along a line. The additive identity 0 is located as a starting point and the multiplicative identity 1 is marked off (usually to the right on a horizontal line). Distances to the right correspond to positive numbers and distances to the left to negative ones. The inequality $<$ indicates which numbers are to the left of others. The abstract properties are as follows.

Axiom 1.2. *Ordered Field Axioms*

A a number system is an ordered field if it satisfies the field Axioms 1.1 *and has a relation* $<$ *that satisfies:*

- *Every pair of numbers* a *and* b *satisfies exactly one of the relations*
$$a = b,\ a < b,\ \text{or}\ b < a$$
- *If* $a < b$ *and* $b < c$*, then* $a < c$*.*
- *If* $a < b$*, then* $a + c < b + c$*.*
- *If* $0 < a$ *and* $0 < b$*, then* $0 < a \cdot b$*.*

These axioms have simple interpretations on the number line. The first order axiom says that every two numbers can be compared; either two numbers are equal or one is to the left of the other.

The second axiom, called transitivity, says that if a is left of b and b is left of c, then a is left of c.

The third axiom says that if a is left of b and we move both by a distance c, then the results are still in the same left-right order.

The fourth axiom is the most difficult abstractly. All the compatibility with multiplication is built from it.

The rational numbers satisfy all these axioms, as do the real and hyperreal numbers. The complex numbers cannot be ordered in a manner compatible with the operations of addition and multiplication.

Definition 1.3. *Absolute Value*
If a is a nonzero number in an ordered field, $|a|$ is the larger of a and $-a$, that is, $|a| = a$ if $-a < a$ and $|a| = -a$ if $a < -a$. We let $|0| = 0$.

Exercise set 1.2

1. *If $0 < a$, show that $-a < 0$ by using the additive property.*
2. *Show that $0 < 1$. (HINT: Recall the exercise that $(-1) \cdot (-1) = 1$ and argue by contradiction, supposing $0 < -1$.)*
3. *Show that $a \cdot a > 0$ for every $a \neq 0$.*
4. *Show that there is no order $<$ on the complex numbers that satisfies the ordered field axioms.*
5. *Prove that if $a < b$ and $c > 0$, then $c \cdot a < c \cdot b$.*
   Prove that if $0 < a < b$ and $0 < c < d$, then $c \cdot a < d \cdot b$.

1.3 The Completeness Axiom

Dedekind's "real" numbers represent points on an ideal line with no gaps.

The number $\sqrt{2}$ is not rational. Suppose to the contrary that $\sqrt{2} = q/r$ for integers q and r with no common factors. Then $2r^2 = q^2$. The prime factorization of both sides must be the same, but the factorization of the squares have an even number distinct primes on each side and the 2 factor is left over. This is a contradiction, so there is no rational number whose square is 2.

A length corresponding to $\sqrt{2}$ can be approximated by (rational) decimals in various ways, for example, $1 < 1.4 < 1.41 < 1.414 < 1.4142 < 1.41421 < 1.414213 < \ldots$. There is no rational for this sequence to converge to, even though it is "trying" to converge. For example, all the terms of the sequence are below $1.41422 < 1.4143 < 1.415 < 1.42 < 1.5 < 2$. Even without remembering a fancy algorithm for finding square root decimals, you can test the successive decimal approximations by squaring, for example, $1.41421^2 = 1.9999899241$ and $1.41422^2 = 2.0000182084$.

It is perfectly natural to add a new number to the rationals to stand for the limit of the better and better approximations to $\sqrt{2}$. Similarly, we could devise approximations to π and make π the number that stands for the limit of such successive approximations. We would like a method to include "all such possible limits" without having to specify the particular approximations. Dedekind's approach is to let the real numbers be the collection of all "cuts" on the rational line.

Definition 1.4. *A Dedekind Cut*

A "cut" in an ordered field is a pair of nonempty sets A and B so that:

- *Every number is either in A or B.*
- *Every a in A is less than every b in B.*

We may think of $\sqrt{2}$ defining a cut of the rational numbers where A consists of all rational numbers a with $a < 0$ or $a^2 < 2$ and B consists of all rational numbers b with $b^2 > 2$. There is a "gap" in the rationals where we would like to have $\sqrt{2}$. Dedekind's "real numbers" fill all such gaps. In this case, a cut of real numbers would have to have $\sqrt{2}$ either in A or in B.

Axiom 1.5. *Dedekind Completeness*

The real numbers are an ordered field such that if A and B form a cut in those numbers, there is a number r such that r is in either A or in B and all other the numbers in A satisfy $a < r$ and in B satisfy $r < b$.

In other words, every cut on the "real" line is made at some specific number r, so there are no gaps. This seems perfectly reasonable in cases like $\sqrt{2}$ and π where we know specific ways to describe the associated cuts. The only drawback to Dedekind's number system is that "every cut" is not a very concrete notion, but rather relies on an abstract notion of "every set." This leads to some paradoxical facts about cuts that do not have specific descriptions, but these need not concern us. Every specific cut has a real number in the middle.

Completeness of the reals means that "approximation procedures" that are "improving" converge to a number. We need to be more specific later, but for example, bounded increasing or decreasing sequences converge and "Cauchy" sequences converge. We will not describe these details here, but take them up as part of our study of limits below.

Completeness has another important consequence, the Archimedean Property Theorem 1.8. We take that up in the next section. The Archimedean Property says precisely that the real numbers contain no positive infinitesimals. Hyperreal numbers extend the reals by including infinitesimals. (As a consequence the hyperreals are not Dedekind complete.)

1.4 Small, Medium and Large Numbers

Hyperreal numbers give us a way to simplify estimation by adding infinitesimal numbers to the real numbers.

We want to have three different intuitive sizes of numbers, very small, medium size, and very large. Most important, we want to be able to compute with these numbers using the same rules of algebra as in high school and separate the 'small' parts of our computation. Hyperreal numbers give us these computational estimates. Hyperreal numbers satisfy three axioms which we take up separately below, Axiom 1.7, Axiom 1.9, and Axiom 2.1.

As a first intuitive approximation, we could think of these scales of numbers in terms of the computer screen. In this case, 'medium' numbers might be numbers in the range -999 to + 999 that name a screen pixel. Numbers closer than one unit could not be distinguished by different screen pixels, so these would be 'tiny' numbers. Moreover, two medium numbers a and b would be indistinguishably close, $a \approx b$, if their difference was a 'tiny' number less than a pixel. Numbers larger in magnitude than 999 are too big for the screen and could be considered 'huge.'

The screen distinction sizes of computer numbers is a good analogy, but there are difficulties with the algebra of screen - size numbers. We want to have ordinary rules of algebra and the following properties of approximate equality. For now, all you should think of is that $\approx$ means 'approximately equals.'

(a) If p and q are medium, so are $p + q$ and $p \cdot q$.
(b) If ε and δ are tiny, so is $\varepsilon + \delta$, that is, $\varepsilon \approx 0$ and $\delta \approx 0$ implies $\varepsilon + \delta \approx 0$.
(c) If $\delta \approx 0$ and q is medium, then $q \cdot \delta \approx 0$.
(d) $1/0$ is still undefined and $1/x$ is huge only when $x \approx 0$.

You can see that the computer number idea does not quite work, because the approximation rules don't always apply. If $p = 15.37$ and $q = -32.4$, then $p \cdot q = -497.998 \approx -498$, 'medium times medium is medium,' however, if $p = 888$ and $q = 777$, then $p \cdot q$ is no longer screen size...

The hyperreal numbers extend the 'real' number system to include 'ideal' numbers that obey these simple approximation rules as well as the ordinary rules of algebra and trigonometry. Very small numbers technically are called infinitesimals and what we shall assume that is different from high school is that there are positive infinitesimals.

Definition 1.6. *Infinitesimal Number*

A number δ in an ordered field is called infinitesimal if it satisfies

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots > \frac{1}{m} > \cdots > |\delta|$$

for any ordinary natural counting number $m = 1, 2, 3, \cdots$. We write $a \approx b$ and say a is infinitely close to b if the number $b - a \approx 0$ is infinitesimal.

This definition is intended to include 0 as "infinitesimal."

Axiom 1.7. *The Infinitesimal Axiom*
The hyperreal numbers contain the real numbers, but also contain nonzero infinitesimal numbers, that is, numbers $\delta \approx 0$, positive, $\delta > 0$, but smaller than all the real positive numbers.

This stands in contrast to the following result.

Theorem 1.8. *The Archimedean Property*
The hyperreal numbers are not Dedekind complete and there are no positive infinitesimal numbers in the ordinary reals, that is, if $r > 0$ is a positive real number, then there is a natural counting number m such that $0 < \frac{1}{m} < r$.

Proof:

We define a cut above all the positive infinitesimals. The set A consists of all numbers a satisfying $a < 1/m$ for every natural counting number m. The set B consists of all numbers b such that there is a natural number m with $1/m < b$. The pair A, B defines a Dedekind cut in the rationals, reals, and hyperreal numbers. If there is a positive δ in A, then there cannot be a number at the gap. In other words, there is no largest positive infinitesimal or smallest positive non-infinitesimal. This is clear because $\delta < \delta+\delta$ and 2δ is still infinitesimal, while if ε is in B, $\varepsilon/2 < \varepsilon$ must also be in B.

Since the real numbers must have a number at the "gap," there cannot be any positive infinitesimal reals. Zero is at the gap in the reals and every positive real number is in B. This is what the theorem asserts, so it is proved. Notice that we have also proved that the hyperreals are not Dedekind complete, because the cut in the hyperreals must have a gap.

Two ordinary real numbers, a and b, satisfy $a \approx b$ only if $a = b$, since the ordinary real numbers do not contain infinitesimals. Zero is the only real number that is infinitesimal.

If you prefer not to say 'infinitesimal,' just say 'δ is a tiny positive number' and think of $\approx$ as 'close enough for the computations at hand.' The computation rules above are still important intuitively and can be phrased in terms of limits of functions if you wish. The intuitive rules help you find the limit.

The next axiom about the new "hyperreal" numbers says that you can continue to do the algebraic computations you learned in high school.

Axiom 1.9. *The Algebra Axiom (Including $<$ rules.)*
The hyperreal numbers are an ordered field, that is, they obey the same rules of ordered algebra as the real numbers, Axiom 1.1 and Axiom 1.2.

The algebra of infinitesimals that you need can be learned by working the examples and exercises in this chapter.

Functional equations like the addition formulas for sine and cosine or the laws of logs and exponentials are very important. (The specific high school identities are reviewed in the main text CD Chapter 28 on High School Review.) The Function Extension Axiom 2.1 shows how to extend the non-algebraic parts of high school math to hyperreal numbers. This axiom is the key to Robinson's rigorous theory of infinitesimals and it took 300 years to discover. You will see by working with it that it is a perfectly natural idea, as hindsight often reveals. We postpone that to practice with the algebra of infinitesimals.

Example 1.2. *The Algebra of Small Quantities*

Let's re-calculate the increment of the basic cubic using the new numbers. Since the rules of algebra are the same, the same basic steps still work (see Example 1.1), except now we may take x any number and δx an infinitesimal.

Small Increment of $f[x] = x^3$

$$\begin{aligned} f[x+\delta x] &= (x+\delta x)^3 = x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^3 \\ f[x+\delta x] &= f[x] + 3x^2\ \delta x + (\delta x[3x+\delta x])\ \delta x \\ f[x+\delta x] &= f[x] + f'[x]\ \delta x + \varepsilon\ \delta x \end{aligned}$$

with $f'[x] = 3x^2$ and $\varepsilon = (\delta x[3x+\delta x])$. The intuitive rules above show that $\varepsilon \approx 0$ whenever x is finite. (See Theorem 1.12 and Example 1.8 following it for the precise rules.)

Example 1.3. *Finite Non-Real Numbers*

The hyperreal numbers obey the same rules of algebra as the familiar numbers from high school. We know that $r+\Delta > r$, whenever $\Delta > 0$ is an ordinary positive high school number. (See the addition property of Axiom 1.2.) Since hyperreals satisfy the same rules of algebra, we also have new finite numbers given by a high school number r plus an infinitesimal,

$$a = r + \delta > r$$

The number $a = r + \delta$ is different from r, even though it is infinitely close to r. Since δ is small, the difference between a and r is small

$$0 < a - r = \delta \approx 0 \qquad \text{or} \qquad a \approx r \quad \text{but} \quad a \neq r$$

Here is a technical definition of "finite" or "limited" hyperreal number.

Definition 1.10. *Limited and Unlimited Hyperreal Numbers*

A hyperreal number x is said to be finite (or limited) if there is an ordinary natural number $m = 1, 2, 3, \cdots$ so that

$$|x| < m.$$

If a number is not finite, we say it is infinitely large (or unlimited).

Ordinary real numbers are part of the hyperreal numbers and they are finite because they are smaller than the next integer after them. Moreover, every finite hyperreal number is near an ordinary real number (see Theorem 1.11 below), so the previous example is the most general kind of finite hyperreal number there is. The important thing is to learn to compute with approximate equalities.

Example 1.4. *A Magnified View of the Hyperreal Line*

Of course, infinitesimals are finite, since $\delta \approx 0$ implies that $|\delta| < 1$. The finite numbers are not just the ordinary real numbers and the infinitesimals clustered near zero. The rules of algebra say that if we add or subtract a nonzero number from another, the result is a different number. For example, $\pi - \delta < \pi < \pi + \delta$, when $0 < \delta \approx 0$. These are distinct finite hyperreal numbers but each of these numbers differ by only an infinitesimal, $\pi \approx \pi + \delta \approx \pi - \delta$. If we plotted the hyperreal number line at unit scale, we could only put one dot for all three. However, if we focus a microscope of power $1/\delta$ at π we see three points separated by unit distances.

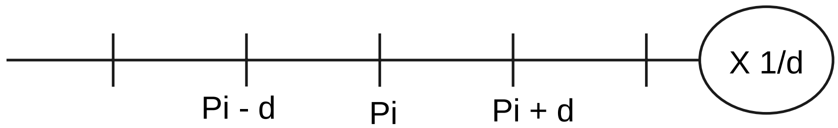

Figure 1.1: Magnification at Pi

The basic fact is that finite numbers only differ from reals by an infinitesimal. (This is equivalent to Dedekind's Completeness Axiom.)

Theorem 1.11. *Standard Parts of Finite Numbers*
Every finite hyperreal number x differs from some ordinary real number r by an infinitesimal amount, $x - r \approx 0$ or $x \approx r$. The ordinary real number infinitely near x is called the standard part of x, $r = \mathrm{st}(x)$.

Proof:

Suppose x is a finite hyperreal. Define a cut in the real numbers by letting A be the set of all real numbers satisfying $a \leq x$ and letting B be the set of all real numbers b with $x < b$. Both A and B are nonempty because x is finite. Every a in A is below every b in B by transitivity of the order on the hyperreals. The completeness of the real numbers means that there is a real r at the gap between A and B. We must have $x \approx r$, because if $x - r > 1/m$, say, then $r + 1/(2m) < x$ and by the gap property would need to be in B.

A picture of the hyperreal number line looks like the ordinary real line at unit scale. We can't draw far enough to get to the infinitely large part and this theorem says each finite number is indistinguishably close to a real number. If we magnify or compress by new number amounts we can see new structure.

You still cannot divide by zero (that violates rules of algebra), but if δ is a positive infinitesimal, we can compute the following:

$$-\delta, \quad \delta^2, \quad \frac{1}{\delta} \qquad \text{What can we say about these quantities?}$$

The idealization of infinitesimals lets us have our cake and eat it too. Since $\delta \neq 0$, we can divide by δ. However, since δ is tiny, $1/\delta$ must be HUGE.

Example 1.5. *Negative infinitesimals*

In ordinary algebra, if $\Delta > 0$, then $-\Delta < 0$, so we can apply this rule to the infinitesimal number δ and conclude that $-\delta < 0$, since $\delta > 0$.

Example 1.6. *Orders of infinitesimals*

In ordinary algebra, if $0 < \Delta < 1$, then $0 < \Delta^2 < \Delta$, so $0 < \delta^2 < \delta$.

We want you to formulate this more exactly in the next exercise. Just assume δ is very small, but positive. Formulate what you want to draw algebraically. Try some small ordinary numbers as examples, like $\delta = 0.01$. Plot δ at unit scale and place δ^2 accurately on the figure.

Example 1.7. *Infinitely large numbers*

For real numbers if $0 < \Delta < 1/n$ then $n < 1/\Delta$. Since δ is infinitesimal, $0 < \delta < 1/n$ for every natural number $n = 1, 2, 3, \ldots$ Using ordinary rules of algebra, but substituting the infinitesimal δ, we see that $H = 1/\delta > n$ is larger than any natural number n (or is "infinitely large"), that is, $1 < 2 < 3 < \ldots < n < H$, for every natural number n. We can "see" infinitely large numbers by turning the microscope around and looking in the other end.

The new algebraic rules are the ones that tell us when quantities are infinitely close, $a \approx b$. Such rules, of course, do not follow from rules about ordinary high school numbers, but the rules are intuitive and simple. More important, they let us 'calculate limits' directly.

Theorem 1.12. *Computation Rules for Finite and Infinitesimal Numbers*

(a) *If p and q are finite, so are $p + q$ and $p \cdot q$.*
(b) *If ε and δ are infinitesimal, so is $\varepsilon + \delta$.*
(c) *If $\delta \approx 0$ and q is finite, then $q \cdot \delta \approx 0$. (finite x infsml = infsml)*
(d) $1/0$ *is still undefined and* $1/x$ *is infinitely large only when* $x \approx 0$.

To understand these rules, just think of p and q as "fixed," if large, and δ as being as small as you please (but not zero). It is not hard to give formal proofs from the definitions above, but this intuitive understanding is more important. The last rule can be "seen" on the graph of $y = 1/x$. Look at the graph and move down near the values $x \approx 0$.

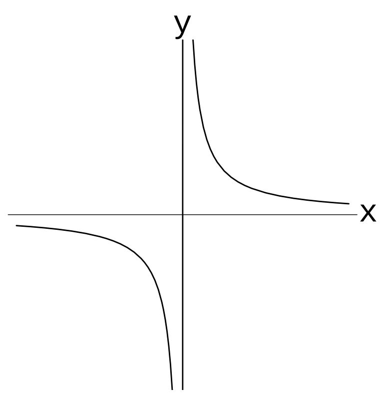

Figure 1.2: $y = 1/x$

Proof:

We prove rule (c) and leave the others to the exercises. If q is finite, there is a natural number m so that $|q| < m$. We want to show that $|q \cdot \delta| < 1/n$ for any natural number n. Since δ is infinitesimal, we have $|\delta| < 1/(n \cdot m)$. By Exercise 1.2.5, $|q| \cdot |\delta| < m \cdot \frac{1}{n \cdot m} = \frac{1}{m}$.

Example 1.8. $y = x^3 \Rightarrow dy = 3x^2\, dx$, *for finite* x

The error term in the increment of $f[x] = x^3$, computed above is

$$\varepsilon = (\delta x[3x + \delta x])$$

If x is assumed finite, then $3x$ is also finite by the first rule above. Since $3x$ and δx are finite, so is the sum $3x + \delta x$ by that rule. The third rule, that says an infinitesimal times a finite number is infinitesimal, now gives $\delta x \times$ finite $= \delta x[3x + \delta x] =$ infinitesimal, $\varepsilon \approx 0$. This justifies the local linearity of x^3 at finite values of x, that is, we have used the approximation rules to show that

$$f[x+\delta x] = f[x] + f'[x]\ \delta x + \varepsilon\ \delta x$$

with $\varepsilon \approx 0$ whenever $\delta x \approx 0$ and x is finite, where $f[x] = x^3$ and $f'[x] = 3\,x^2$.

Exercise set 1.4

1. *Draw the view of the ideal number line when viewed under an infinitesimal microscope of power $1/\delta$. Which number appears unit size? How big does δ^2 appear at this scale? Where do the numbers δ and δ^3 appear on a plot of magnification $1/\delta^2$?*

2. Backwards microscopes or compression
Draw the view of the new number line when viewed under an infinitesimal microscope with its magnification reversed to power δ (not $1/\delta$). What size does the infinitely large number H (HUGE) appear to be? What size does the finite (ordinary) number $m = 10^9$ appear to be? Can you draw the number H^2 on the plot?

3. $y = x^p \Rightarrow dy = p\,x^{p-1}\ dx,\ p = 1, 2, 3, \ldots$
For each $f[x] = x^p$ below:

(a) *Compute $f[x+\delta x] - f[x]$ and simplify, writing the increment equation:*

$$\begin{aligned} f[x+\delta x] - f[x] &= f'[x]\cdot\delta x + \varepsilon\cdot\delta x \\ &= [\textit{term in } x \textit{ but not } \delta x]\delta x + [\textit{observed microscopic error}]\delta x \end{aligned}$$

Notice that we can solve the increment equation for $\varepsilon = \dfrac{f[x+\delta x] - f[x]}{\delta x} - f'[x]$

(b) *Show that $\varepsilon \approx 0$ if $\delta x \approx 0$ and x is finite. Does x need to be finite, or can it be any hyperreal number and still have $\varepsilon \approx 0$?*

(1) *If $f[x] = x^1$, then $f'[x] = 1x^0 = 1$ and $\varepsilon = 0$.*
(2) *If $f[x] = x^2$, then $f'[x] = 2x$ and $\varepsilon = \delta x$.*
(3) *If $f[x] = x^3$, then $f'[x] = 3x^2$ and $\varepsilon = (3x + \delta x)\delta x$.*
(4) *If $f[x] = x^4$, then $f'[x] = 4x^3$ and $\varepsilon = (6x^2 + 4x\delta x + \delta x^2)\delta x$.*
(5) *If $f[x] = x^5$, then $f'[x] = 5x^4$ and $\varepsilon = (10x^3 + 10x^2\delta x + 5x\delta x^2 + \delta x^3)\delta x$.*

4. Exceptional Numbers and the Derivative of $y = \dfrac{1}{x}$

(a) *Let $f[x] = 1/x$ and show that*

$$\frac{f[x+\delta x] - f[x]}{\delta x} = \frac{-1}{x(x+\delta x)}$$

(b) *Compute*

$$\varepsilon = \frac{-1}{x(x+\delta x)} + \frac{1}{x^2} = \delta x \cdot \frac{1}{x^2(x+\delta x)}$$

(c) *Show that this gives*

$$f[x+\delta x] - f[x] = f'[x]\cdot\delta x + \varepsilon\cdot\delta x$$

when $f'[x] = -1/x^2$.

(d) *Show that $\varepsilon \approx 0$ provided x is NOT infinitesimal (and in particular is not zero.)*