Conformal Geometry of Discrete Groups and Manifolds

by

Boris N. Apanasov

Walter de Gruyter · Berlin · New York 2000

Author

Boris N. Apanasov
Department of Mathematics
University Oklahoma
601 Elm Avenue, PHSC 423
Norman, OK 73019-0315
USA
apanasov@math.ou.edu

Mathematics Subject Classification 2000:
51-01, 51-02, 57-01, 57-02; 11Fxx, 20-XX, 22E40, 30C65, 30F40, 32Gxx, 52C25, 53Axx, 53Cxx, 55Nxx, 55Rxx, 57Mxx, 57Nxx

Key words:
Geometric Structures on Manifolds, Discrete Groups, Conformal Geometry, Hyperbolic Geometry, Deformations of Structures, Teichmüller Spaces

Printed on acid-free paper which falls within the guidelines of the ANSI to ensure permanence and durability.

Library of Congress – Cataloging-in-Publication Data

Apanasov, B. N. (Boris Nikolaevich)
Conformal geomety of discrete groups and manifolds / by Boris N. Apanasov.
p. cm. – (De Gruyter expositions in mathematics, ISSN 0938-6572 ; 32)
Includes bibliographical references and index.
ISBN 3 11 014404 2 (alk. paper)
1. Conformal geometry. 2. Discrete groups. 3. Manifolds (Mathematics) I. Title. II. Series.
QA609 .A63 2000
516.3′5–dc21 00-043017

Die Deutsche Bibliothek – Cataloging-in-Publication Data

Apanasov, Boris N.:
Conformal geometry of discrete groups and manifolds / by Boris N. Apanasov. – Berlin ; New York : de Gruyter, 2000
(De Gruyter expositions in mathematics ; 32)
ISBN 3-11-014404-2

Typesetting using the author's TEX files: I. Zimmermann, Freiburg.
Printing: WB-Druck GmbH & Co., Rieden/Allgäu. Binding: Lüderitz & Bauer-GmbH, Berlin.
Cover design: Thomas Bonnie, Hamburg.

To Tatiana, Anton and Nikolay

Preface

One of the most beautiful results in classical complex analysis that has a great appeal to geometry is the solution by F. Klein and H. Poincaré of the uniformization problem for multivalent analytic functions and (consequently) Riemann surfaces. It states that each conformal structure on a Riemann surface is induced by one of the three classical geometries: Euclidean, spherical or hyperbolic (Lobachevskian), that is this structure is represented by a Riemannian metric on the surface which has a constant (zero, positive or negative) curvature.

According to Felix Klein's Erlangen program of 1872, geometry is the study of the properties of a space which are invariant under a group of transformations. A geometry in Klein's sense is thus a pair (X, G) where X is a manifold and G is a Lie group transitively acting on X. Due to the Klein–Poincaré geometrization theorem, the Euclidean, spherical and hyperbolic geometries are the most important ones in dimension two. However, they are all particular cases of the more general *conformal geometry*, that is the $(S^2, \text{Möb}(2))$-geometry, where $\text{Möb}(n)$ is the group of conformal (Möbius) transformations of the n-dimensional sphere S^n. This is not a Riemannian geometry. A conformal structure on a manifold M is the same as a conformal class of Riemannian metrics, each locally conformally equivalent to a flat metric. In dimension three, due to Thurston's geometrization, many 3-manifolds admit conformal structures, although relatively simple ones might not (among the eight possible 3-geometries, nontrivial closed solvable and nilpotent manifolds are examples of this). Generally, conformal geometries naturally appear at infinity for negatively curved Riemannian geometries. Moreover, due to M. Gromov's [5, 6] geometric approach to infinite groups, conformal geometry invents new fruitful methods in combinatorial group theory.

The main goal of our book is to present the first systematic study of conformal geometry of n-manifolds, as well as its Riemannian counterparts (in particular, hyperbolic geometry). A unifying theme is the discrete holonomy groups of the corresponding geometric structures, which also involves algebra and dynamics. However, we do not pay much attention to 2-dimensional geometries covered by many classical and recent books (see, for example, Casson–Bleiler [1], Beardon [4], Ford [1], Kra [3], Maskit [12]). Also, this book does not cover conformal geometries that appear at infinity for noncompact symmetric spaces with variable sectional curvature. Nevertheless we indicate some relationship to those geometries and provide the necessary references.

Regarding hyperbolic geometry, some recent books may be useful for the reader as a source of alternative approaches and references: Benedetti–Petronio [1], Ratcliffe [1] (with rich historical notes) and Apanasov [36]. We have minimized the unavoidable overlap in the covered results on hyperbolic manifolds by using those books, especially the last one (which is an English edition of the 1983 Russian book Apanasov [15]), as a source for preliminary results and constructions. Additionally, the enormous expansion in journal literature on conformal geometry of manifolds and the new important results obtained in the last decades years have allowed us to point out new connections and perspectives in this field and to illustrate various aspects of the theory. In addition to formal proofs we also indicate some intuitive approaches, which emphasize the ideas behind the constructions. This is complemented by a large number of concrete examples (continuing the book by S. Krushkal', B. Apanasov and N. Gusevskii [3]) and figures which both use and support the reader's geometric imagination and make the matter more transparent. We have tried to make the book as complete as possible, although the choice of topics obviously reflects our personal preferences.

Our interest in this area started in the "golden years" of mathematics in Novosibirsk Akademgorodok in the seventies and eighties, when the author worked in a remarkable geometry/topology group that included A. D. Aleksandrov, P. P. Belinskii, V. Goldstein, N. Gusevskii, S. Krushkal', V. Marenich, A. Mednykh, I. Nikolaev, Yu. G. Reshetnyak, V. Toponogov and several doctoral students: D. Derevnin, V. Chueshev, M. Kapovich, E. Klimenko, G. Lyan, L. Potyagailo, A. Tetenov, A. Vesnin, S. Vodopyanov, I. Zhuravlev and others. This provided a perfect environment for our research and for advanced graduate courses we taught at the Novosibirsk State University and at the Sobolev Institute of Mathematics in the Academy of Sciences. These courses were continued in graduate courses we taught at the Universitat Autonoma de Barcelona (Spain) and at the University of Oklahoma in Norman, USA. This book is based on those courses, and it should be accessible to advanced graduate students in either mathematics or theoretical physics. In particular, the first three chapters (which make the book self-contained) are addressed to those graduate students who are approaching the subject for the first time. These chapters may be used as a text for a graduate class. The book quickly introduces these students to up-to-date problems. To the second type of readers, mature mathematicians working in other fields and theoretical physicists, this book gives new knowledge and understanding of conformal geometry on manifolds and the conformal action of fundamental groups. To the experts, the book presents some new material published for the first time.

It is our deep pleasure to thank our colleagues and friends with whom we discussed the subject for a long time. In addition to those mentioned above, a debt of gratitude for a series of valuable remarks is owed to Francis Bonahon, Dubravko Ivanšić, Anatoly Fomenko, Michael Gromov, Yoshinobu Kamishima, Ann Chi Kim, Ravi

Kulkarni, Kyung Bai Lee, Gregory Margulis, Sergei Matveev, Alexander Mednykh, Darryl McCullough, Jose-Maria Montesinos Amilibia, Walter Neumann, Vladislav Nikulin, Ken'ichi Ohshika, Masahico Saito, Larry Siebenmann, Alexander Starkov, Bill Thurston, Andrei Vesnin, Ernest B. Vinberg, Oleg Viro and Chengbo Yue.

Our special thanks are due to Nikolai Saveliev for discussions and help during the writing this book. Finally, we are deeply indebted to the Walter de Gruyter Publishing Company, especially to Dr. Manfred Karbe, for encouraging us to write this book.

Norman, Spring 2000 *Boris Apanasov*

Table of Contents

Preface vii

1. Geometric Structures 1

1.1. (X, G)-structures on manifolds 1
1.2. Conformal geometry on the sphere 4
1.3. The hyperbolic space $\mathbb{H}^n$ 11
1.4. Lie subgroups of the Möbius group 19
1.5. Structure developments and holonomy homomorphisms 22
1.6. The eight 3-dimensional geometries 25
1.7. Four-dimensional geometries 31
1.8. Geometry of orbifolds 33
Notes 39

2. Discontinuous Groups of Homeomorphisms 40

2.1. Convergence groups 40
2.2. Group action on the discontinuity set 54
2.3. Fundamental domains 62
2.4. Convex polyhedra and reflection groups 77
2.5. Discrete group action on the limit set 86
Notes 100

3. Basics of Hyperbolic Groups and Manifolds 101

3.1. Margulis's Lemma and splittings of hyperbolic manifolds 101
3.2. Injectivity radius of hyperbolic manifolds 107
3.3. Thin cusp submanifolds 111
3.4. Precisely invariant horoballs 117
3.5. Group action on the set of horoballs 121
3.6. Convex hull constructions 123
3.7. Tessellations of manifolds by ideal hyperbolic polyhedra 129

3.8. Hyperbolic arithmetics 130
3.9. Arithmetic groups generated by reflections 133
3.10. Non-arithmetic groups of Gromov and Piatetski–Shapiro 140
3.11. Fibonacci manifolds 144
Notes 151

4. Geometrical Finiteness 153

4.1. Classical finiteness for planar Kleinian groups 153
4.2. Geometrical finiteness in higher dimensions 156
4.3. Equivalent definitions of geometrical finiteness 166
4.4. Geometrically finite ends and coverings 175
4.5. Geometry of tessellations 178
4.6. Cayley graphs and geometric isomorphisms of discrete groups 191
4.7. Geometrical finiteness for discontinuity set components 204
Notes 215

5. Kleinian Manifolds 217

5.1. Basic topology related to Kleinian manifolds 217
5.2. Topological aspects of combination theorems 225
5.3. Universal groups and Poincaré Conjecture 241
5.4. Ends of Kleinian manifolds, their compactification and Ahlfors's Conjecture 244
5.5. Kleinian n-manifolds and hyperbolic cobordisms 256
5.6. Finiteness problems for Kleinian n-manifolds 276
Notes 288

6. Uniformization 289

6.1. Classical uniformization 290
6.2. Modern concepts of uniformization 291
6.3. Hyperbolization of manifolds and hyperbolic volumes 308
6.4. Uniformizable conformal structures 324
6.5. Conformal uniformization of "flat" connected sums 335
6.6. Conformal uniformization of Seifert manifolds 342
6.7. Torus sums of conformal structures 350
6.8. Canonical Riemannian metric on conformal manifolds 355
Notes 377

7. Theory of Deformations 379

7.1. Deformations of geometric structures 379
7.2. Rigidity of hyperbolic structures 392
7.3. Quasi-Fuchsian structures: bendings 408
7.4. Quasi-Fuchsian structures: cone deformations 425

7.5. Bendings along surfaces with boundaries 434
7.6. Global properties of deformation spaces 441
Notes 457

Bibliography 461

Index 513

Chapter 1

Geometric Structures

In addition to purely topological methods in the study of manifolds, last two decades results and especially Thurston's work [1–8] have shown that geometry also plays an important role in low-dimensional topology. The basic aim of this chapter is to introduce (following Thurston [1]) the important concept of geometric (Riemannian and sub-Riemannian) structures on manifolds and its generalization for manifolds with singularities (so-called orbifolds). We will also discuss various 'nice' geometries which arise in dimensions three and four, as well as the relationship between geometric and topological properties of manifolds carrying those geometries. This discussion will be continued in Chapter VI.

§1. (X, G)-structures on manifolds

Let M be a topological Hausdorff space with a countable basis. One calls M a topological *n-manifold* (n-dimensional manifold) if each point $x \in M$ has a neighborhood U homeomorphic to the Euclidean space $\mathbb{R}^n = \varphi(U)$. If in this condition, the homeomorphic image $\varphi(U)$ may be either the Euclidean space $\mathbb{R}^n$ or its closed half-space $\{x \in \mathbb{R}^n : x_n \geq 0\}$, we arrive at the definition of a *manifold with boundary*. Here the set of points $x \in M$ not having neighborhoods homeomorphic to the Euclidean space forms the boundary ∂M, which is a manifold of dimension $(n-1)$. A compact manifold without boundary is called closed. In such a way, a topological n-manifold is specified by its atlas $\{(U_i, \varphi_i)\}_{i \in \mathcal{I}}$ consisting of local charts $\varphi_i\colon U_i \to \mathbb{R}^n$ where, for $U_i \cap U_j \neq \emptyset$, changes of charts $\varphi_i \varphi_j^{-1}$ are homeomorphisms defined on subdomains in $\mathbb{R}^n$. M is called a *smooth manifold* if, for any charts (U_1, φ_1) and (U_2, φ_2) on M in the chosen atlas with $U_1 \cap U_2 \neq \emptyset$, the changes of charts $\varphi_1 \circ \varphi_2^{-1}$ are smooth.

Now let M be a smooth manifold. For each $x \in M$, we denote by $T_x(M)$ the *tangent space* of the manifold M at x (an n-dimensional topological vector space). Consider the set $T(M) = \bigcup_{x \in M} T_x(M)$ and the natural projection $p\colon T(M) \to M$ such that $p^{-1}(x) = T_x(M)$. The triple $\{T(M), M, p\}$ is called a *tangent fiber bundle of the manifold* M, while the set $p^{-1}(x)$ is called a *fiber* in the fiber bundle over $x \in M$.

A *Riemannian manifold* is said to be a pair (M, g) consisting of a smooth manifold M and a smooth mapping $g\colon T(M) \to \mathbb{R}$, which is a positive-definite quadratic form in each fiber $T_x(M)$. The Riemannian manifold (M, g) is a metric space whose metric is generated by those quadratic forms. That is, for points $y_0, y_1 \in M$,

$$\rho(y_0, y_1) = \inf_{\gamma} \operatorname{length}(\gamma) = \inf_{\gamma} \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t))}dt\,, \tag{1.1}$$

where the lower bound is taken over all continuously differentiable curves γ in M joining points y_0 and y_1, i.e., $\gamma(0) = y_0$ and $\gamma(1) = y_1$. The function $\rho\colon M \times M \to \mathbb{R}$ has all the properties of a metric.

A curve $\gamma \subset M$ is described as being the *shortest* if its length is the least among all curves with the same ends. It is clear from (1.1) that the curve γ with ends y_0 and y_1 will be the shortest if and only if its length is $\rho(y_0, y_1)$. A curve $\gamma \subset M$ is called a *geodesic* if each of its points has a neighborhood such that each arc of the curve γ in this neighborhood is the shortest one.

Considering three kinds of curvatures for Riemannian manifolds (sectional curvature, Ricci curvature and scalar curvature), we will be mainly concerned with the sectional curvature which has a geometric description making it easier to handle. One may define the sectional curvature in a point p of a Riemannian manifold M with respect to a 2-subspace $V \subset T_pM$ as the curvature at p of the oriented Riemann surface obtained as the image of a small neighborhood $U(0) \subset V$ under the exponential mapping.

There is another way to add more structure to a manifold. Namely, instead of considering an atlas on M, one can think about M as if it were composed of pieces of $\mathbb{R}^n$, glued together by the homeomorphisms $g_{ij} = \varphi_i\varphi_j^{-1}$. Denoting this set of homeomorphisms by $\mathcal{G}$, we see that it should satisfy the following obvious conditions which transform $\mathcal{G}$ to a pseudo-group of local homeomorphisms between open sets in $\mathbb{R}^n$:

1) a restriction g_0 of any element $g \in \mathcal{G}$ to an open set in its domain is an element of $\mathcal{G}$;
2) a composition $g \circ h$ of any two elements $g, h \in \mathcal{G}$ (if defined) is an element of $\mathcal{G}$;
3) the inverse element for $g \in \mathcal{G}$ is an element of $\mathcal{G}$; and
4) if $D = \bigcup_i D_i \subset \mathbb{R}^n$ and $g\colon D \to D'$ is a local homeomorphism such that $g_i = g|_{D_i} \in \mathcal{G}$ for all i, then $g \in \mathcal{G}$.

Manifolds M, obtained by gluing together pieces of $\mathbb{R}^n$ by means of local homeomorphisms from a pseudo-group $\mathcal{G}$, are called *$\mathcal{G}$-manifolds*.

Specifying a pseudo-group $\mathcal{G}$ in this definition, one can add more structure on a $\mathcal{G}$-manifold M. For example, we arrive at the notion of a PL-manifold if $\mathcal{G}$ is the pseudo-group of local piecewise-linear homeomorphisms in $\mathbb{R}^n$, and at the notion of a C^r-manifold (a C^r-smooth manifold, for $r \geq 1$) if $\mathcal{G}$ is the pseudo-group of local

C^r-diffeomorphisms in $\mathbb{R}^n$. Furthermore, a very important class of $\mathcal{G}$-manifolds can be obtained by using the following pseudo-groups $\mathcal{G}$.

Let us fix a manifold X and a group G of self-homeomorphisms of X and consider a pseudo-group $\mathcal{G}$ consisting of all restrictions of elements of G to open subsets in X. A manifold M with such $\mathcal{G}$-structure is represented as composed of pieces of X by means of elements of the pseudo-group $\mathcal{G}$ associated with the group G. We call such $\mathcal{G}$-manifolds (X, G)-*manifolds* or manifolds *modeled on* (X, G)-geometry.

Here, we use the term "geometry" in the sense of F. Klein, meaning by geometry of the pair (X, G) those properties of X that are left invariant under the group G action. Sometimes, this "invariant" approach to studying geometry is equivalent to the classical approach which can be used to study Euclidean and non-Euclidean geometries as well as to the Riemannian geometry approach (see Theorem 1.14).

Example 1.1 (Affine torus). Let $X = \mathbb{R}^n$ and $G = \mathrm{Aff}(\mathbb{R}^n)$ be the group of affine transformations in $\mathbb{R}^n$. Then a $(\mathbb{R}^n, \mathrm{Aff}(\mathbb{R}^n))$-manifold is an *affine* n-manifold. In particular, an affine structure may be defined on two-dimensional torus $T^2 = S^1 \times S^1$ by gluing the opposite sides of a quadrilateral which needs not to be a parallelogram. This gluing is performed by means of two affine transformations in the plane (see Figure 1), i.e., by elements of the affine pseudo-group, which are restrictions of these affine transformations on corresponding open subsets in $\mathbb{R}^2$ containing sides of the quadrilateral. However, we shall see in what follows that, for a non-parallelogram case, a torus with such structure is not a complete affine manifold.

Example 1.2 (Euclidean torus). One can introduce a Euclidean structure on *n-dimensional torus* $T^n = S^1 \times \cdots \times S^1$ whose fundamental group is a free Abelian group of rank n. In fact, this torus T^n may be obtained by gluing the opposite sides of an n-dimensional parallelepiped, (Figure 2). The gluing mappings may be chosen to be Euclidean translations, and thus a complete Euclidean structure is introduced on the torus T^n.

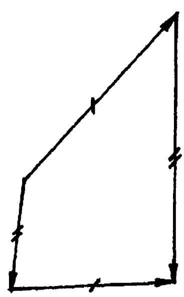

Figure 1

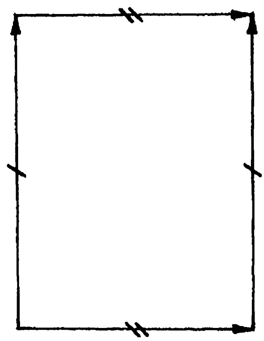

Figure 2