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INTRODUCTION TO INTERSECTION THEORY IN ALGEBRAIC GEOMETRY

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Contents

Preface

These lectures are designed to provide a survey of modern intersection theory in algebraic geometry. This theory is the result of many mathematicians' work over many decades; the form espoused here was developed with R. MacPherson.

In the first two chapters a few episodes are selected from the long history of intersection theory which illustrate some of the ideas which will be of most concern to us here. The basic construction of intersection products and Chern classes is described in the following two chapters. The remaining chapters contain a sampling of applications and refinements, including theorems of Verdier, Lazarsfeld, Kempf, Laksov, Gillet, and others.

No attempt is made here to state theorems in their natural generality, to provide complete proofs, or to cite the literature carefully. We have tried to indicate the essential points of many of the arguments. Details may be found in **[16]**.

I would like to thank R. Ephraim for organizing the conference, and C. Ferreira and the AMS staff for expert help with preparation of the manuscript.

1. Intersections of Hypersurfaces

1.1. Early history (Bézout, Poncelet). A most basic question in intersection theory is to describe the intersection of several algebraic hypersurfaces in n-space, i.e., the common solutions of several polynomials in n variables. The ancients certainly knew about the possible intersections of lines and conics in the plane, and they also knew that rational solutions of two quadric equations in three variables behaved like solutions of one cubic equation in two variables [**61**].

We do not know who first observed that two plane curves of degrees p and q should intersect in pq points. By 1680 Newton [**48**] had developed an elimination theory for two such equations. This produced a *resultant*, which was a polynomial in one variable of degree pq whose solutions gave an abscissa of the intersection points of the two curves. The corresponding construction and assertion for n equations in n variables were made in 1764 by Bézout [**5**, **6**]. Bézout's treatment was entirely algebraic, although he briefly interpreted his result for $n = 2$ and $n = 3$: the number of intersections of two plane curves (or three surfaces in space) is *at most* the products of their degrees.

By referring to the resultants, which are polynomials in one variable, one can also discuss the possibilities of *nonreal* solutions, *asymptotic* solutions, and *multiple* solutions. As geometry developed, the first two of these situations were subsumed by considering intersections of hypersurfaces $H_1, \ldots, H_n$ in complex projective space $\mathbf{P}^n_{\mathbf{C}}$. Now we assign an *intersection multiplicity*

$$i(P) = i(P, H_1 \cdots H_n)$$

to a point P of the intersection $\bigcap H_i$; if the H_i do not meet transversally at P, this multiplicity will be greater than one.

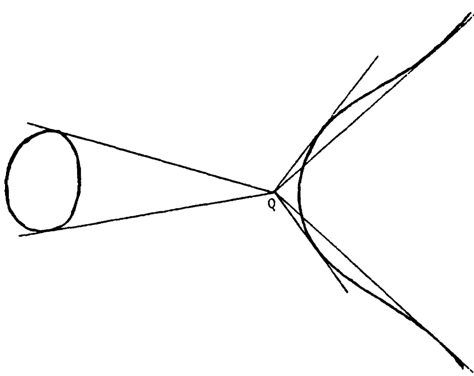
Although there was little early discussion of this multiplicity, the governing *principle of continuity* was well understood, at least since Poncelet [**51**]. If the H_i vary in families $H_i(t)$, with $H_i(0) = H_i$, and $P_1(t), \ldots, P_r(t)$ are the points of the general intersection $\bigcap H_i(t)$ which approach P as $t \to 0$, then

$$i(P, H_1 \cdots H_n) = \sum_{j=1}^{r} i\big(P_j(t), H_1(t) \cdots H_n(t)\big).$$

Varying the H_i so that the $H_i(t)$ meet transversally, this determines the multiplicity $i(P, H_1 \cdots H_n)$.

In all the above discussion, it is assumed that the intersection of the hypersurfaces is a finite set, or at least that P is an isolated point of $\bigcap H_i$.

1.2. Class of a curve (Plücker). An important early application of Bézout's theorem was for the calculation of the *class* of a plane curve C, i.e., the number of tangents to C through a given general point Q:



Equivalently, the class of C is the degree of the dual curve $C^{\vee}$. If $F(x, y, z)$ is the homogeneous polynomial defining C and $Q = (a : b : c)$, then the *polar curve* C_Q is defined by

$$F_Q(x, y, z) = aF_x + bF_y + cF_z,$$

where $F_x = \partial F(x, y, z)/\partial X$, F_y, F_z are partial derivatives. This is defined so that a nonsingular point P of C is on C_Q exactly when the tangent line to C at P (defined by $XF_x(P) + YF_y(P) + ZF_z(P) = 0$) passes through Q. One checks that C meets C_Q transversally at P if P is not a flex on C, so

$$\text{class}(C) = \#C \cap C_Q = \deg C \deg C_Q = n(n-1),$$

if n is the degree of C, and C is nonsingular.

If C has singular points, however, they are always on $C \cap C_Q$, so they must contribute. For example, if P is an ordinary node (resp. cusp) and Q is general, then

$$i(P, C \cdot C_Q) = 2 \quad (\text{resp. } i(P, C \cdot C_Q) = 3).$$

This gives the first Plücker formula [50]

$$n(n-1) = \text{class}(C) + 2\delta + 3\kappa,$$

if C has degree n, δ ordinary nodes, κ ordinary cusps, and no other singularities

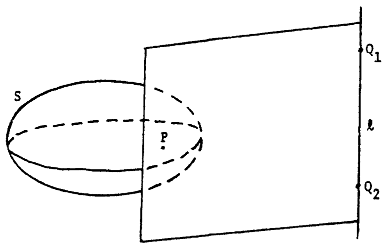
1.3. Degree of a dual surface (Salmon). In 1847 Salmon [53] made a similar study of surfaces. If $S \subset \mathbf{P}^3$ is a surface, the degree of the dual (or "reciprocal") surface $S^{\vee}$ is the number of points $P \in S$ such that the tangent plane to S at P

contains a given general line l. (This number is one of the projective characters of S, now called the *second class* of S.)

For a point $Q \in \mathbf{P}^3$, let S_Q be the *polar surface* of S with respect to Q: if $F(x, y, z, w)$ defines S and $Q = (a : b : c : d)$, then $aF_x + bF_y + cF_z + dF_w$ defines S_Q. Taking two points Q_1, Q_2 on l, one sees as before that a nonsingular point P of S is on $S_{Q_1} \cap S_{Q_2}$ if and only if the tangent plane to S at P contains l. Thus for S nonsingular of degree n, and Q_1, Q_2 general,

$$\deg(S^\vee) = \#S \cap S_{Q_1} \cap S_{Q_2} = n(n-1)^2.$$

As before, all singular points of S are contained in $S \cap S_{Q_1} \cap S_{Q_2}$. If P is an isolated singular point of S, its contribution to the total $n(n-1)^2$ is the intersection multiplicity $i(P, S \cdot S_{Q_1} \cdot S_{Q_2})$. For example, the contribution of an ordinary double point is two, so $\deg(S^\vee) = n(n-1)^2 - 2\delta$ if S has δ ordinary double points.

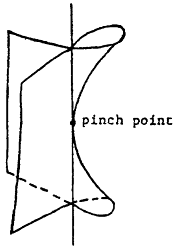


If S is singular along a curve C, however, a new phenomenon occurs, a problem of *excess intersection*: how to compute the contribution of C to the total intersection $n(n-1)^2$, so that $n(n-1)^2$ diminished by this contribution, and by contributions of other singular points, yields $\deg(S^\vee)$. Salmon initiates a study of the contribution of a curve C to the intersection of three surfaces in space when C is a component of their intersection. For example, if C is a line, he gives its contribution as $m + n + p - 2$, where m, n, p are the degrees of the surfaces. Salmon justifies this by saying that the answer must be independent of the choice of surfaces of given degrees, and then he calculates it directly in the degenerate case when the first is the union of a plane containing C and a general surface of degree $m - 1$. This surface meets the other two surfaces in $(m-1)np$ points, $m - 1$ of which are on the line C. The plane meets the other two surfaces in curves of degrees $n - 1$ and $p - 1$ in addition to C; these curves meet in $(n-1)(p-1)$ points. The total number of points of intersection outside C is therefore

$$(m-1)np - (m-1) + (n-1)(p-1) = mnp - (m+n+p-2),$$

as asserted. In case C is a double line on the first surface, he calculates its contribution as $m + 2n + 2p - 4$ by working out the case where this surface is the union of two surfaces containing C.

If C is a double line on a surface S of degree n, this analysis predicts $5n - 8$ as the contribution of C to the intersection of S with S_{Q_1} and S_{Q_2}. However, as Salmon points out, there are special points on C, called *pinch points* (or "cuspidal" points), where the two tangent planes to S coincide.



If C is the line $x = y = 0$, and S is the surface $Uy^2 + Vxy + Wy^2 = 0$, then these pinch points are the intersections of C with the surface $V^2 = 4UW$, so there are $2n - 4$ pinch points on S. Thus C, together with its pinch points, diminishes the degree of $S^\vee$ by $(5n - 8) + (2n - 4) = 7n - 12$. For example, a cubic with a double line (e.g. $y^2 = zx^2 + x^3$) has a dual surface of degree three.

Salmon also considers more general curves. If C is a complete intersection of surfaces of degrees a and b, and C is a component of intersection of three ~~surfaces~~ of degrees m, n, and p, then he finds that the contribution of C to the total number of mnp is $ab(m + n + p - (a + b))$. Concluding this remarkable paper, he deduces that if such C is an r-fold curve on a surface S, then it diminishes the degree of the dual by

$$ab\left[(r - 1)(3r + 1)n - r^2(r - 1)(a + b) - 2r(r - 1)\right].$$

1.4. The problem of five conics. Problems of excess intersection arise frequently in enumerative problems. The famous problem of the number of plane conics tangent to five given conics in general position is a typical example of this. A plane conic is defined by a quadratic polynomial $ax^2 + by^2 + cxy + dx + ey + f$, unique up to multiplication by a nonzero scalar, so the space of conics can be identified with $\mathbf{P}^5$. One checks that the condition to be tangent to a fixed nonsingular conic is described by a hypersurface of degree six in $\mathbf{P}^5$. The desired conics are then represented by the points in the intersection of five such hypersurfaces $H_1 \cap \cdots \cap H_5$. There are not $6^5 = 7776$ such conics, however, as originally thought by Steiner and others. Indeed, the Veronese surface $V \cong \mathbf{P}^2$ of conics which are double lines is contained in $\bigcap H_i$, and one can show (cf. §4 below) that the contribution of V to the intersection is actually 4512, which leaves 3264, the actual number of (nonsingular) conics tangent to five given conics in general position.

Note that the conics tangent to a fixed line form a quadric hypersurface in $\mathbf{P}^6$. Given five general lines, the Veronese contributes 31 to the predicted intersection number 2^5 for the five quadrics. Since everyone knew that there is only one nonsingular conic tangent to five general lines (by duality, for example), it is curious that these false answers were proposed when the lines are replaced by curves of higher degree.

In spite of the clear exposition of the importance of excess intersections in enumerative geometry by Salmon and Cayley, such considerations played little role in the great development of enumerative geometry at the hands of Chasles, de Jonquières, Schubert, Halphen, Zeuthen, and others. For one thing, they avoided writing equations for varieties and, especially, for parameter spaces. In general, however, their work can be interpreted as calculating intersections on appropriate spaces so that the intersections become proper. Often these spaces are blow-ups of the naive spaces, which amounts to adding structure to degenerate figures. For example, a classical approach to the space of conics amounts to working on the space of *complete* conics, which is the blow-up $\tilde{\mathbf{P}}^5$ of $\mathbf{P}^5$ along the Veronese; in this model a point in the exceptional divisor corresponds to a double line together with a pair of points on the line. The proper transforms of the hypersurfaces H_i then meet properly on $\tilde{\mathbf{P}}^5$ outside the exceptional divisor, and once one knows an appropriate "intersection ring" for $\tilde{\mathbf{P}}^5$ one may calculate their intersection.

The same approach works for quadrics of arbitrary dimension. The beautiful study of complete quadrics was initiated by Schubert, who found many enumerative formulas. The rigorous construction of these parameter spaces and their intersection rings has been carried out by Semple and Tyrell, with modern re-examination by Vainsencher, Laksov, and Lazarsfeld. Realizing the spaces as orbit spaces of suitable group actions, by Demazure and by De Concini and Procesi, has led to a clearer understanding of their structure.

1.5. A dynamic formula (Severi, Lazarsfeld). In general, if $H_1, \ldots, H_n$ are arbitrary hypersurfaces in $\mathbf{P}^n$, with $d_i = \deg(H_i)$, Severi [58] proposed to assign numbers $i(Z)$ to certain distinguished subvarieties Z of the intersection locus $H_1 \cap \cdots \cap H_n$, so that

$$\sum i(Z) = d_1 \cdots d_n.$$

Each irreducible component of $\bigcap H_i$ should be distinguished, and each isolated point should be assigned its intersection multiplicity. In general, as in Salmon's examples, there may also be imbedded distinguished varieties. Severi's dynamic procedure, corrected and completed by Lazarsfeld [40], can be summarized as follows. If F_i is a homogeneous equation for H_i, consider deformations $H_i(t)$ of H_i defined by homogeneous polynomials $F_i + tG_i + t^2G_i' + \cdots$. For a given subvariety Z of $\bigcap H_i$, let $j(Z)$ be the number of points of $\bigcap H_i(t)$ which approach Z as $t \to 0$, for a *generic* deformation; in fact, $j(Z)$ of the points will approach Z for any deformation for which the first order parts $(G_1, \ldots, G_n)$ belong to a certain open set U_Z of the space of n-tuples of polynomials of degrees $d_1, \ldots, d_n$. For any point P set $i(P) = j(P)$. Only finitely many points will have $i(P) \neq 0$. For an irreducible curve C, set

$$i(C) = j(C) - \sum_{P \in C} i(P),$$

so $i(C)$ is the number of points that generically approach C, but not any particular point on C. Inductively,

$$i(Z) = j(Z) - \sum i(V),$$

the sum over all proper irreducible subvarieties V of Z. Then $\sum i(Z) = d_1 \cdots d_n$, which achieves the desired decomposition.

We will later see a *static* construction of this decomposition, which is also valid in contexts where such deformations are unavailable. It should be emphasized, however, that in spite of the existence of a rigorous general theory, and some explicit formulas, the actual computation of the contributions $i(Z)$ remains a difficult problem.

For plane curves, following Segre [55], Lazarsfeld gives the following answer. If $H_i = D_i + E$, where D_1 and D_2 meet properly, and P is a point in E, let G_i be generic as above, let A_i be equations for D_i, and let F be the curve defined by $A_1 G_2 - A_2 G_1$. Then

$$i(P) = i(P, E \cdot F) + i(P, D_1 \cdot D_2).$$

For example, if $H_1 = 2L_1 + L_2$, $H_2 = L_1 + 2L_2$, with L_1, L_2 lines meeting at a point P, then the Segre-Lazarsfeld formula shows that

$$i(P) = i(L_1) = i(L_2) = 3.$$

1.6. Algebraic multiplicity, resultants. For an isolated point P in the intersection of hypersurfaces $H_1, \ldots, H_n$ in $\mathbf{P}^n$, a modern *static* definition of the intersection multiplicity is

$$i(P, H_1 \cdots H_n) = \dim_{\mathbf{C}} \mathcal{O}_P / (f_1, \ldots, f_n),$$

where $\mathcal{O}_P$ is the local ring of $\mathbf{P}^n$ at P, and f_i is a local equation for H_i in $\mathcal{O}_P$. If P is the origin in $\mathbf{C}^n \subset \mathbf{P}^n$, $\mathcal{O}_P$ is the localization of $\mathbf{C}[X_1, \ldots, X_n]$ at the maximal ideal $(X_1, \ldots, X_n)$. Or one may replace $\mathcal{O}_P$ by its completion $\mathbf{C}[[X_1, \ldots, X_n]]$, or by the ring $\mathbf{C}\{X_1, \ldots, X_n\}$ of convergent power series. This algebraic construction of intersection multiplicity dates from Macaulay [42].

Let us verify the agreement of this definition with that obtained from elimination theory, at least for plane curves. Suppose the curves are defined by polynomials $f(x, y)$ and $g(x, y)$, and the two curves do not meet at infinity on the y-axis. Thus we may assume

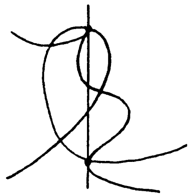
$$f(x, y) = a_0(x) y^n + a_1(x) y^{n-1} + \cdots + a_n(x)$$

with $a_0(0) \neq 0$. Let $A = \mathbf{C}[x]_{(x)}$ be the local ring of the x-axis at the origin. Then $A[y]/(f)$ is an A-algebra which is a free A-module of rank n, and one may construct the *resultant* $r = R(f, g)$ in A by

$$r = \det\left(A[y]/(f) \xrightarrow{\cdot g} A[y]/(f)\right).$$

(It is a formal exercise, left to the reader, to show that this agrees with the usual definition, as in [**60**].)

We must show that the order of vanishing of r at $x = 0$ is equal to the sum of the intersection numbers of the two curves at all points P on the y-axis:



Now $A[y]/(f, g)$ is finite dimensional over $\mathbf{C}$, so it is a direct sum of its localizations $\mathcal{O}_P/(f, g)$, where P varies over the points on the y-axis on both curves. Therefore

$$\sum_P i(P) = \dim_{\mathbf{C}} A[y]/(f, g).$$

Since the order of vanishing of r at $x = 0$ is $\dim_{\mathbf{C}} A/(r)$, the equation to be proved is

$$\dim_{\mathbf{C}} A[y]/(f, g) = \dim_{\mathbf{C}} A/(r).$$

This is a special case of an important algebraic fact:

LEMMA. *Let A be a one-dimensional Noetherian local domain, M a finitely generated free A-module and ϕ: $M \to M$ an A-homomorphism. Then*

$$\operatorname{length}_A(M/\phi(M)) = \operatorname{length}_A(A/(\det(\phi))).$$

The *length* of an A-module N is d if there is a chain of submodules $N = N_0 \supset N_1 \supset \cdots \supset N_d = 0$, where N_i/N_{i+1} is isomorphic to the residue field of A. In case A contains a subfield K which maps isomorphically to its residue field, then $\operatorname{length}_A N = \dim_K N$.

When A is a discrete valuation ring, the lemma is an exercise in elementary divisors. For the general case see [**16**, A2.6].

2. Multiplicity and Normal Cones

2.1. Geometric multiplicity. A *subvariety* X of $\mathbf{C}^N$ is defined by a prime ideal $I(X)$ in $\mathbf{C}[X_1, \ldots, X_N]$. The *coordinate ring* $\Gamma(X)$ is the residue ring

$$\Gamma(X) = \mathbf{C}[X_1, \ldots, X_n]/I(X).$$

A (closed) *subscheme* Z of X is determined by an ideal $I = I(Z)$ of $\Gamma(X)$, which is a *subvariety* if $I(Z)$ is prime. In this case the *local ring of X at Z* is the localization of $\Gamma(X)$ at $I(Z)$, and is denoted $\mathcal{O}_{Z,X}$.

If Z is a subscheme of X, the *irreducible components* of Z are the subvarieties of X corresponding to the minimal prime ideals of $\Gamma(X)$ which contain $I(Z)$. If V is such a component, the *geometric multiplicity of V in Z* is defined to be the length of the Artinian ring

$$\mathcal{O}_{V,Z} = \mathcal{O}_{V,X}/I(Z)\mathcal{O}_{V,X}.$$

The *cycle* of Z, denoted $[Z]$, is defined to be the formal sum

$$[Z] = \sum_{i=1}^{r} m_i[V_i],$$

where $V_1, \ldots, V_r$ are the irreducible components of Z, and m_i is the geometric multiplicity of V_i in Z. For example, if $X = \mathbf{C}^n$ and Z is the scheme-theoretic intersection of n hypersurfaces which meet properly, then

$$[Z] = \sum i(P)[P],$$

the sum over the points P in Z, with $i(P)$ the intersection number described in §1.6.

For an arbitrary variety X, subschemes Z are defined by ideal sheaves $\mathcal{I} = \mathcal{I}(Z)$. On any affine open $U \subset X$ which meets Z, $\mathcal{I}$ is given by an ideal in the coordinate ring of U, which is prime if Z is a subvariety. The local ring of X along V, and the geometric multiplicity of a component V of Z can be defined using any such U.

2.2. Hilbert polynomials. A subscheme Z of $\mathbf{P}^N$ is defined by a homogeneous ideal $I = I(Z)$ in $\mathbf{C}[X_0, \ldots, X_N]$. If $\mathbf{C}[X_0, \ldots, X_N]_t$ denotes the homogeneous polynomials of degree t, such an ideal I is the direct sum of its intersections I_t with $\mathbf{C}[X_0, \ldots, X_N]_t$. Two homogeneous ideals define the same subscheme when their homogeneous pieces are the same for all but finitely many t. The *Hilbert polynomial* of Z is the polynomial $P_Z(t)$ such that

$$P_Z(t) = \dim_{\mathbf{C}}(\mathbf{C}[X_0, \ldots, X_N]_t/I_t)$$

for all sufficiently large t. Indeed, one shows (cf. [**30**, §1.7; or **57**]) that the right side is a polynomial of degree equal to the dimension of Z, for $t \gg 0$. If $n = \dim(Z)$, one may define the *degree* of Z, $\deg(Z)$, to be the coefficient of $t^n/n!$ in $P_Z(t)$, i.e.

(i) $$P_Z(t) = \deg(Z)t^n/n! + \text{lower terms}.$$

It also follows that if $[Z] = \Sigma m_i[V_i]$ is the cycle of Z, then

(ii) $$\deg(Z) = \sum_{\dim(V_i)=n} m_i \deg(V_i).$$

If V is a subvariety of $\mathbf{P}^N$, and H is a hypersurface of $\mathbf{P}^N$ not containing V, then

(iii) $$\deg(V \cap H) = m \deg(V).$$

It will later become clear that this definition of $\deg(V)$ agrees with the geometric notion of counting intersections of V with complementary linear spaces. In fact, we shall have no need for Hilbert polynomials, although they have played an important role in the modern algebraic development of multiplicity.

2.3. A refinement of Bézout's theorem. The elementary facts about degree in the preceding section, together with an important join construction, allow a simple proof of the following proposition. A stronger result will appear later when more intersection theory is available.

PROPOSITION. *Let $V_1, \ldots, V_s$ be subvarieties of $\mathbf{P}^N$, and let $Z_1, \ldots, Z_r$ be the irreducible components of $V_1 \cap \cdots \cap V_s$. Then*

$$\sum_{i=1}^{r} \deg(Z_i) \leqslant \prod_{j=1}^{s} \deg(V_j).$$

PROOF. By a simple induction, one may assume $s = 2$. Construct the *ruled join* $J = J(V_1, V_2)$ in $\mathbf{P}^{2N+1}$ as follows. Let $X_0, \ldots, X_N$, $Y_0, \ldots, Y_N$ be homogeneous coordinates on $\mathbf{P}^{2N+1}$. Let $\mathbf{P}_1^N$ (resp. $\mathbf{P}_2^N$) be the linear subspace of $\mathbf{P}^{2N+1}$ defined by the vanishing of all Y_i (resp. all X_i). Identifying $\mathbf{P}_i^N$ with $\mathbf{P}^N$, one has $V_i \subset \mathbf{P}_i^N$. Let J be the union of all lines from points of V_1 to points of V_2. Algebraically, the homogeneous coordinate ring of J is simply the tensor product of the homogeneous coordinate rings of V_1 and V_2. One verifies that

(i) $$\deg(J) = \deg(V_1)\deg(V_2).$$

Let L be the linear subspace of $\mathbf{P}^{2N+1}$ defined by $X_i = Y_i$, $0 \leqslant i \leqslant N$. Then $L = \mathbf{P}^N$ and

(ii) $$L \cap J = V_1 \cap V_2.$$

Thus we are reduced to the case where one of the varieties being intersected is a linear subspace.

Since a linear subspace is an intersection of hyperplanes, one is further reduced inductively to the case where one of the varieties, say V_1, is a hyperplane. In this case, either $V_1 \supset V_2$ and the proposition holds with equality, or $[V_1 \cap V_2] = \sum_{i=1}^r m_i[Z_i]$, where the Z_i are the irreducible components of $V_1 \cap V_2$, and by (ii) and (iii) of §2.2 (for any hypersurface V_1 not containing V_2),

$$\sum m_i \deg(Z_i) = \deg(V_1)\deg(V_2).$$

2.4. Samuel's intersection multiplicity. Suppose $H_1, \ldots, H_n$ are hypersurfaces in an n-dimensional variety V, and P is an isolated point of $\cap H_i$. Let $A = \mathcal{O}_{P,V}$ be the local ring of V along P, and assume each H_i is defined by one element f_i in A. Let $I = (f_1, \ldots, f_n)$. Then A/I is finite dimensional over $\mathbf{C}$, and if P is a nonsingular point of V, one may use $\dim_{\mathbf{C}} A/I$ to give a workable definition of the intersection multiplicity $i(P, H_1 \cdots H_n)$ as in §1. The following is a standard example of the failure of this definition in general.

EXAMPLE. Let V be the image of the mapping $\phi\colon \mathbf{C}^2 \to \mathbf{C}^4$ defined by $\phi(s,t) = (s^4, s^3t, st^3, t^4)$, let P be the origin, and let H_1 and H_2 be the hypersurfaces of V defined by the coordinates x_1 and x_4 respectively. By varying H_1 and H_2, the principle of continuity requires that the intersection multiplicity is 4. However, one calculates that the ideal of V is generated by $x_1x_4 - x_2x_3$, $x_1^2x_3 - x_2^3$, $x_2x_4^2 - x_3^3$, and $x_2^2x_4 - x_3^2x_1$, from which it follows that $\dim_{\mathbf{C}} A/(x_1, x_4) = 5$.

Samuel [54] defines the multiplicity $i(P) = i(P, H_1 \cdots H_n)$ to be the coefficient of $t^n/n!$ in the Hilbert-Samuel polynomial

$$P(t) = \dim_{\mathbf{C}}(A/I^t) = i(P)t^n/n! + \text{lower terms} \tag{i}$$

for $t \gg 0$. To see that $\dim(A/I^t)$ is a polynomial of degree n in t, for $t \gg 0$, one may proceed as follows. Let $\Lambda = A/I$ and consider the surjection of graded rings

$$\Lambda[X_1, \ldots, X_n] \to \bigoplus_{t=0}^{\infty} I^t/I^{t+1} \tag{ii}$$

which maps X_i to the image of f_i in I/I^2. The kernel of this homomorphism is a homogeneous ideal which defines a subscheme $\mathbf{P}(C)$ of projective $(n-1)$-space $\mathbf{P}_\Lambda^{n-1}$ over Λ. (Those who feel uncomfortable with projective space over a ring such as Λ may realize $\mathbf{P}(C)$ in $\mathbf{P}^{n-1} \times V$, since Λ is a residue ring of A.) This scheme $\mathbf{P}(C)$ is the *projective normal cone* to $\cap H_i$ in V. We shall discuss normal cones in succeeding sections. Here we shall use the fact that $\mathbf{P}(C)$ has pure dimension $n-1$, so its Hilbert polynomial has the form

$$P_{\mathbf{P}(C)}(t) = \dim_{\mathbf{C}} I^t/I^{t+1} = i(P)t^{n-1}/(n-1)! + \cdots \tag{iii}$$

for $t \gg 0$. A simple calculation shows that this definition of $i(P)$ is the same as that in (i). However, since $\mathbf{P}(C) \subset \mathbf{P}_\Lambda^{n-1}$, the only component of $\mathbf{P}(C)$ is the underlying variety $\mathbf{P}_{\mathbf{C}}^{n-1}$ of $\mathbf{P}_\Lambda^{n-1}$ and, therefore,

$$[\mathbf{P}(C)] = i(P)[\mathbf{P}_{\mathbf{C}}^{n-1}] \tag{iv}$$

defines the multiplicity $i(P)$ without reference to Hilbert functions. In addition, since $\mathbf{P}(C) \subset \mathbf{P}_{\Lambda}^{n-1}$, and

$$[\mathbf{P}_{\Lambda}^{n-1}] = \dim_{\mathbf{C}}(\Lambda)[\mathbf{P}_{\mathbf{C}}^{n-1}],$$

it follows that

$$\text{(v)} \qquad i(P) \leqslant \dim_{\mathbf{C}}(\Lambda) = \dim_{\mathbf{C}} A/(f_1,\dots,f_d).$$

We see also that equality holds in (v) if the morphism (ii) is an isomorphism. This is related to the important notion of a regular sequence.

DEFINITION. A sequence of elements $f_1,\dots,f_d$ in the maximal ideal of a local ring A is a *regular sequence* if f_1 is a non-zero-divisor in A, and if, for $i = 2,\dots,d$, the image of f_i in $A/(f_1,\dots,f_{i-1})$ is a non-zero-divisor. (This is equivalent to asserting that the Koszul complex

$$0 \to \Lambda^d(A^d) \to \Lambda^{d-1}(A^d) \to \cdots \to A^d \to A$$

defined by $f_1,\dots,f_d$ is exact, giving a resolution of A/I. In fact, the multiplicity $i(P)$ may also be defined to be the alternating sum of the dimensions of the homology groups of this complex, cf. [**57**].)

The *dimension* of a local ring A is the length n of a maximal chain of prime ideals $P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n \subsetneqq A$. If A is the local ring of a variety V along a subvariety W, the dimension of A is the codimension of W in V. The ring A is *Cohn-Macaulay* if its maximal ideal contains a regular sequence of $\dim(A)$ elements. For example if P is a nonsingular point of V, then $\mathcal{O}_{P,V}$ is Cohen-Macaulay.

The following lemma contains the main facts from commutative algebra that we will need. For proofs, see [**38** or **57**].

LEMMA. (a) *If A is Cohen-Macaulay, a sequence $f_1,\dots,f_d$ of elements in the maximal ideal of A is a regular sequence if and only if*

$$\dim(A/(f_1,\dots,f_d)) = \dim(A) - d.$$

(b) *Let $f_1,\dots,f_d$ be a regular sequence in a local ring A, and let $I = (f_1,\dots,f_d)$. Then the canonical homomorphism*

$$A/I[X_1,\dots,X_d] \to \bigoplus_{t=0}^{\infty} I^t/I^{t+1},$$

which takes X_i to the image of f_i in I/I^2, is an isomorphism. Moreover, the kernel of the canonical surjection

$$A[X_1,\dots,X_d] \to \bigoplus_{t=0}^{\infty} I^t$$

is generated by the elements $f_iX_j - f_jX_i$, $1 \leqslant i < j \leqslant d$.

For example, with notation as at the beginning of this section, if $\mathcal{O}_{P,V}$ is Cohen-Macaulay, it follows that

$$i(P, H_1 \cdots H_n) = \dim_{\mathbf{C}} \mathcal{O}_{P,V}/(f_1,\dots,f_n),$$

i.e., Samuel's sophisticated multiplicity agrees with the naive multiplicity of §1.

2.5. Normal cones. If W is a subscheme of an affine variety V, defined by an ideal I in the coordinate ring A of V, the *normal cone* $C = C_W V$ to W in V is defined to be

$$C = \operatorname{Spec}\left(\bigoplus I^t/I^{t+1}\right).$$

The isomorphism of the coordinate ring of W with $A/I = I^0/I^1$ determines a morphism $p_C\colon C \to W$, called the *projection*, and a closed imbedding $s_C\colon W \to C$, called the *zero section*, with $p_C \circ s_C = \mathrm{id}_W$. If $f_d, \ldots, f_d$ generate I, the canonical surjection of $A/I[X_1, \ldots, X_d]$ onto $\oplus I^t/I^{t+1}$ determines a closed imbedding of C in $W \times \mathbf{C}^d$:

$$\begin{tikzcd} C \arrow[rr, hook] \arrow[dr, shift left, "p_C"] & & W \times \mathbf{C}^d \arrow[dl, shift right, "\mathrm{pr}"'] \\ & W \arrow[ul, shift left, "s_C"] \arrow[ur, shift right, "\mathrm{zero}"'] & \end{tikzcd}$$

If $f_1, \ldots, f_d$ is a regular sequence, it follows from the preceding lemma that $C = W \times \mathbf{C}^d$. In general, since C is defined by a homogeneous ideal, it is a subcone of $W \times \mathbf{C}^d$, i.e., C is invariant under multiplication by $\mathbf{C}^*$ on the fibres $\mathbf{C}^d$.

In spite of the marvelous brevity of this algebraic definition of normal cone, its geometry is not so simple. Considerable study, beginning with [**32**], has been devoted to the case where W is a nonsingular subvariety. For example, if $W = P$ is a point, then $C_P V$ is the *tangent cone* to V at P; if V is a hypersurface in $\mathbf{C}^d$, and P is the origin, one may check that $C_P V$ is the hypersurface in $\mathbf{C}^d$ defined by the leading homogeneous term of an equation for V. However, as is evident from the preceding section, the normal cones of interest for intersection theory are usually defined by ideals which are not prime ideals, i.e. W is a subscheme of V, but not usually a subvariety. There has been extensive recent study of associated graded rings $\oplus I^t/I^{t+1}$ in commutative algebra; one hopes that useful criteria for identifying the irreducible components of C, with their multiplicities, may emerge.

The *projective normal cone* $\mathbf{P}(C) = \mathbf{P}(C_W V)$ is defined by

$$\mathbf{P}(C) = \operatorname{Proj}\left(\bigoplus I^t/I^{t+1}\right).$$

In concrete terms, if generators for I are chosen as above, $\mathbf{P}(C)$ is the subscheme of $W \times \mathbf{P}^{d-1}$ defined by the same equations that define C in $W \times \mathbf{C}^d$.

A closely related and equally important construction is that of the *blow-up* of a variety V along a subscheme W. This is a variety $\tilde{V} = \mathrm{Bl}_W V$, together with a proper morphism $\pi\colon \tilde{V} \to V$, satisfying:

(i) The inverse image scheme $E = \pi^{-1}(W)$ is a Cartier divisor on $\tilde{V}$, called the *exceptional divisor*: at each $Q \in E$, the ideal $I\mathcal{O}_{Q,\tilde{V}}$ has one generator.

(ii) E is isomorphic to $\mathbf{P}(C_W V)$, and the mapping from E to W induced by π is the projection from $\mathbf{P}(C)$ to W:

$$\begin{tikzcd} \mathbf{P}(C) = E \arrow[r, hook] \arrow[d] & \tilde{V} = \mathrm{Bl}_W V \arrow[d, "\pi"] \\ W \arrow[r, hook] & V \end{tikzcd}$$

(iii) The induced mapping from $\tilde{V} - E$ to $V - W$ is an isomorphism.

A quick definition of $\mathrm{Bl}_W V$ is

$$\mathrm{Bl}_W V = \mathrm{Proj}\left(\bigoplus_{t=0}^{\infty} I^t \right),$$

the mapping π determined by the isomorphism of A with I^0. If $f_1, \ldots, f_d$ generate I, $\mathrm{Bl}_W V$ is the subvariety of $V \times \mathbf{P}^{d-1}$ defined by the kernel of the canonical homomorphism from $A[X_1, \ldots, X_d]$ onto $\oplus I^t$. In case $f_1, \ldots, f_d$ is a regular sequence, $\mathrm{Bl}_W V$ is defined by the equations $f_i X_j - f_j X_i$, $i < j$, by the lemma of §2.4. In general, $\mathrm{Bl}_W V$ is the closure of the graph of the morphism from $V - W$ to $\mathbf{P}^{d-1}$ defined by $(f_1 : \ldots : f_d)$.

Note that, since A is a domain, $\oplus I^t$ is also a domain, so $\mathrm{Bl}_W V$ is a variety. The identification of $E = \pi^{-1}(W)$ with $\mathbf{P}(C)$ follows from the canonical isomorphism

$$\left(\bigoplus I^t \right) \otimes_A A/I = \bigoplus I^t / I^{t+1}.$$

Over the subvariety of $\mathbf{P}^{d-1}$ where the coordinate X_i is not zero, W is defined by the equation f_i, since $f_j = (X_j / X_i) f_i$.

One important consequence of this construction is that each irreducible component of $E = \mathbf{P}(C)$ has dimension $d - 1$. Indeed, E is locally defined by one equation in the d-dimensional variety $\tilde{V}$, and any such subscheme has pure codimension one.

The above constructions globalize to the case of an arbitrary proper closed subscheme W of an arbitrary variety V. If $\mathcal{I}$ is the ideal sheaf of W in V, they are written

$$C_W V = \mathrm{Spec}\left(\bigoplus \mathcal{I}^t / \mathcal{I}^{t+1} \right),$$
$$\mathbf{P}(C_W V) = \mathrm{Proj}\left(\bigoplus \mathcal{I}^t / \mathcal{I}^{t+1} \right),$$
$$\mathrm{Bl}_W V = \mathrm{Proj}\left(\bigoplus \mathcal{I}^t \right).$$

They may be constructed by covering V by affine neighborhoods, over which the preceding constructions apply, and gluing over the overlaps.

In case the imbedding of W in V is a *regular imbedding*, i.e., local equations for the ideal of W in V form a regular sequence in local rings of V, then $C_W V$ is a vector bundle, called the *normal bundle* to W in V, and also denoted $N_W V$. If V and W are nonsingular, this agrees with the definition of $N_W V$ as the quotient of tangent bundles:

$$0 \to T_W \to T_V|_W \to N_W V \to 0.$$

When D is an effective Cartier divisor, on a variety X, $N_D X$ is the restriction to D of the associated line bundle $\mathcal{O}_X(D)$ on X. If $E = \mathbf{P}(C)$ is the exceptional divisor on the blow-up $\tilde{V}$ of a variety V along a subscheme W, then

$$N_E \tilde{V} = \mathcal{O}_{\tilde{V}}(E)|_E = \mathcal{O}_C(-1)$$

is also the dual line bundle to the *canonical line bundle* $\mathcal{O}_C(1)$ on $\mathbf{P}(C)$.