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Atanu Sengupta
Tapan Kumar Pal

Fuzzy Preference Ordering of Interval Numbers in Decision Problems

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Preface

In conventional mathematical programming, coefficients of problems are usually determined by the experts as crisp values in terms of classical mathematical reasoning. But in reality, in an imprecise and uncertain environment, it will be utmost unrealistic to assume that the knowledge and representation of an expert can come in a precise way.

The wider objective of the book is to study different real decision situations where problems are defined in inexact environment. Inexactness are mainly generated in two ways – (1) due to imprecise perception and knowledge of the human expert followed by vague representation of knowledge as a DM; (2) due to hugeness and complexity of relations and data structure in the definition of the problem situation. We use interval numbers to specify inexact or imprecise or uncertain data. Consequently, the study of a decision problem requires answering the following initial questions –

How should we

- compare and define preference ordering between two intervals?
- interpret and deal inequality relations involving interval coefficients?
- interpret and make way towards the goal of the decision problem?

The present research work consists of two closely related fields: first-one approaches towards defining a generalized preference ordering scheme for interval attributes. We have suggested two indices, *viz.*, Acceptability Index and Fuzzy Preference Ordering for interval attributes. But, before that a detailed comparative study on the existing ordering indices was necessary to identify the lacunae. Here we liked to incorporate possibly all relevant and even the most recent literatures up to 2008. Naturally, this part of our contribution, spanned more or less within Chapter 1 – 4, has taken considerable room in this book.

The next related field approaches to deal with some issues having application potential in many areas of decision making. Here we tried to develop mathematical description of relationships using interval arithmetic or re-structure some optimization models in inexact environment using interval numbers. Development of working algorithm for these models has also been attempted.

This book consists of nine chapters including the introductory first chapter, which gives a brief review of the literature and basic intention of this study. Our research contribution constitutes Chapter 2 to Chapter 8. The last chapter gives the chapter-wise summary and indicates the direction and scope in brief of the future research.

It is a great opportunity to this publication under the series ‘Studies in Fuzziness and Soft Computing’ by Physica-Verlag (A Springer-Verlag company). We wish to thank the publishers.

We sincerely thank Professor Ajoy Kumar Ray (Head, SMST & Professor, E&ECE Dept., I.I.T. Kharagpur) for his enthusiastic encouragement and support in the preparation of this manuscript. We are fortunate to have constant support and invaluable suggestions from Dr. Debjani Chakraborty (Dept. of Maths., I.I.T. Kharagpur) in all phases of our work. We deeply thank her. We are greatly indebted to Professor Debasish Mondal (Dept. of Economics, VU) for his sincere inspiration and support. We are indebted to Professor R.N. Jana and Professor M.M. Pal of our department for their cordial support in many ways. We also like to extend our sincere thanks to all of our friends and colleagues who supported us in numerous ways in preparing this book.

Atanu Sengupta
Tapan Kumar Pal

Contents

Chapter 1

Introduction

"There must be an ideal world, a sort of mathematician's paradise where everything happens as it does in textbooks."

-- Bertrand Russell (1872-1970)

In our quest for such an ideal world with utmost precision and certainty we often try to replicate the pervasiveness of the real world into a formal model that aims to be a kind of abstraction of domain and the forces and dynamics of the environment.

But in reality, formal (mathematical) models happen to be based on precise information, certain assumptions, crisp hypotheses and well-defined theories, which are logically sound but very little, can mimic reality. For example, formal (mathematical) models used in decision aid are usually characterized by multiple parameters. A situation with *precise information* occurs when the Decision Makers (DMs) are able to indicate a precise value for each parameter. However, there are always many difficulties for obtaining precise values for all the parameters:

- the performance of each action on each criterion may be unknown at the time of the analysis – it may result from arbitrariness in constructing parts of the model or from the aggregation of several aspects having impact on different criteria, and it may result from a measuring instrument or from a statistical tool (which usually involves certain or uncertain extent of error);

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- not all but many parameters, which define relations between the effects and consequences while framing parts of the model, generally have seldom objective existence – they are mostly reflected by the DMs' subjective perception and opinion, which the DM may find difficult to express in so-called 'mathematical precision' and an objective measure of which may change from man-to-man in time-to-time;
- in either case, the result of a quest for numerical precision and certainty is neither real nor accurate. A decision situation related to human aspect, in fact, has only a little to do with the absolute attributes – certainty and precision – which are not present in our cognition, perception, reasoning and thinking. There are so many things which can only be defined by vague and ambiguous predicates and as a result it has been increasingly clear that formal modelling of a real decision situation does not reflect the pervasiveness of human perception, cognition and mutual interaction with the outside world (Gupta (1988)).

I would like to quote from the works of Sir Karl Popper (1974) on the philosophy of science in this context:

> Both precision and certainty are false ideals. They are impossible to attain, and therefore dangerously misleading if they are uncritically accepted as guides. The quest for precision is analogous to the quest for certainty, and both should be abandoned... one should never try to be more precise than the problem situation demands…

Quite a similar note can be found in the words of one of the greatest scientists of all times, Albert Einstein (1879-1955), who says:

> "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

As a matter of fact, in quest of realistic model building and analyses, the representation and manipulation of these *inexact*, *vague*, *fuzzy*, *ambiguous* or *imprecise* information and relations that abound in real world situations become one of the major concerns to the *new-era mathematicians* (the proponents of *fuzzy-ism*, after Zadeh (1962, 1965)'s pioneering call) and since the late 1980's this ensued in an highly demanding field of research, viz., *computing with words*. Different researchers used different distinct expressions to name this type of information:

- *imprecise information* is used in the works of Athanassopoulos & Podinovski (1997), Miettinen & Salminen (1999),
- the term *incomplete information* is used by Weber (1987),

- *partial information* is used by Hazen (1986) and also by Deng (1982) to define his *grey systems theory*.
- *Poor information* is used by Bana e Costa & Vincke (1995).
- *nonpoint information*, a new designation of such information is given by Dias (web document).

This concept of *imprecise information* generalizes the concepts of multiple states of affairs and the concepts of interval parameters. At a given stage of the decision process, there may be a set of the multiple acceptable combinations of parameter values or may be a set of combinations compatible with the available imprecise information (Dias (web document)). This data set may be discrete or continuous. In the latter case the set is defined by the bounds for the parameter values what we commonly call as numerical interval information.

1.1. Conventional distinction between uncertainty and imprecision

Prior to Zadeh's pioneering work on fuzzy sets, the probability theory based on Boolean Logic was an important tool to deal with uncertainties (or randomness) of real events and activities. Fuzzy set theory provided a valuable conceptual tool for dealing with non-stochastic imprecision or so-called vague concepts. Both the notions cover two different types of ignorance and are dealt with two different types of theories with ideas of their own.

Probabilistic uncertainty relates to events that have a well-defined, unambiguous meaning. In probability theory, the notion of probability is required only to quantify our ignorance (uncertainty) about which particular case is going to be observed. We perfectly know the set and the distribution of the elements (in terms of the properties of the underlying population) in the set but we do not know which particular case will be observed in a random experiment. The notion of probability is used to express a forecast about belongingness or not belongingness of an event (that is going to be observed) in a set that is perfectly well known. Probability theory is based on classical set theory and on two-valued logic, e.g. true-or-false or yes-or-no statements; probability theory assesses whether an event will occur (Klir & Folger (1988), Kosko (1990)).In contrast, the fuzzy set theory is based on the notion that the individual elements of a set are well known but the set is not well-defined or ill-defined in such a way that one simply can not decide by 'yes' or 'no' about the belongingness of an element in the particular set. Fuzzy set theory is based on multi-valued logic (Zimmermann (1991), McNeill & Freiberger (1993), Klir & Yuan (1995), Chakraborty (1995)).

The probabilities represent uncertainties but not the degrees of partial truths, which are used for the description of imprecision by degrees of membership. If imprecision is the state of nature of a situation and the resulting uncertainty is possibilistic rather than probabilistic, then the situation is said to be fuzzy (Buckley (1987), (Babad & Berliner (1994)). Probability theory and fuzzy set theory are obviously not the alternative concepts, but are well accepted complementary tools to describe many specific instances of uncertainty and imprecision (Chakraborty (1995), Mukaidono (2001)). Zadeh (1978) has used the term *vagueness* to designate *fuzziness* and *ambiguity*, however, Dubois & Prade (1980)'s *fuzziness* is *vagueness* but it differs from *ambiguity* and *generality*.

1.2. Imprecise data representation: Preliminaries on Fuzzy set and genesis of Interval Numbers as imprecise data

The concept of a fuzzy set was introduced by Zadeh (1965) to represent or manipulate data and information possessing non-statistical uncertainties.

In classical set theory, a subset A of a set X can be defined by its characteristic function χ_A as a mapping from the elements of X to the elements of the set $\{0, 1\}$,

$$\chi_A : X \to \{0, 1\}.$$

This mapping may be represented as a set of ordered pairs, with exactly one ordered pair for each element of X. The truth or falsity of the statement "x is in A" is determined by the ordered pair $(x,\ \chi_A(x))$. The statement is true if $\chi_A = 1$ and the statement is false if $\chi_A = 0$.

Similarly, a fuzzy subset $\tilde{A}$ in a universe of discourse X is characterized by its membership function $\mu_{\tilde{A}}$ as a mapping from the elements of X to the values of the interval $[0,\ 1]$, such that,

$$\mu_{\tilde{A}} : X \to [0, 1].$$

The degree to which the statement "x is in fuzzy set $\tilde{A}$" is true is denoted by the set of ordered pairs $\left(x,\ \mu_{\tilde{A}}(x)\right)$, where $\mu_{\tilde{A}}(x)$ is interpreted as the degree of membership of element x in fuzzy set $\tilde{A}$. A fuzzy set defines the concept of a set with unsharp (imprecise or fuzzy) boundary between the membership and non-membership of its elements (Kaufmann & Gupta (1991)).

Definition: The *support* of the fuzzy subset $\tilde{A}$, denoted as, $\text{supp}(\tilde{A})$ is the crisp subset of X having all elements with non-zero membership grades in $\tilde{A}$. Notationally stated,

$$\text{supp}(\tilde{A}) = \left\{ x \in X \,\middle|\, \mu_{\tilde{A}}(x) > 0 \right\}.$$

Synonyms of support can be the degree/extent of fuzziness or the fuzzy spread (Prodanovic & Simonovic (2002)).

Definition: The height of a fuzzy set is the largest membership grade attained by any element in that set. A fuzzy set $\tilde{A}$ in the universe of discourse X is called *normalized* when the height of $\tilde{A}$ is equal to 1 (Klir & Yuan (1995)).

Definition: A fuzzy set $\tilde{A}$ in the universe of discourse X is convex, if and only if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \quad \geq \quad \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$$

for all x_1, x_2 in X and all $\lambda \in [0,1]$, where *min* denotes the minimum operator (Klir & Yuan (1995)).

Definition: The α-level set (or α-cut) of a fuzzy set $\tilde{A}$ of X is a non-fizzy set (or a crisp interval) denoted by $[\tilde{A}]^{\alpha}$ and defined as

$$[\tilde{A}]^{\alpha} = \left\{ x : \mu_{\tilde{A}}(x) \geq \alpha, \ \ x \in X \right\}, \text{ if } \alpha \in (0,1]$$

$$= \text{cl}(\text{supp}(\tilde{A})), \text{ if } \alpha = 0,$$

where, $\text{cl}(\text{supp}(\tilde{A}))$ denotes the closer of support of $\tilde{A}$ (Dubois & Prade (1980)).

Definition: A *fuzzy number* (or more generally, *fuzzy quantity*) $\tilde{N}$ is a convex and normalized fuzzy subset of the real line $\Re$ (Dubios & Prade (1980), Kaufmann & Gupta (1991)).

In many situations we often summarize numeric information, as for example, around Rs. 5000, near zero, about $10^{\circ}C$, about 15 – 20%, possibly not less than 2000 units. These sorts of numerically transmittable data are not precise or crisp in terms of classical mathematical reasoning but are very meaningful in terms of human communication, perception and reasoning. These imprecise data could be the examples of what are called *fuzzy numbers*.

The membership function of a fuzzy number $\tilde{N}$ has the following properties:

(i) $\mu_{\tilde{N}}(x) = 0$, outside of some interval $[a, d]$;

(ii) There are real numbers b and c, $a \le b \le c \le d$ such that $\mu_{\tilde{N}}(x)$ is monotone increasing on the interval $[a, b]$ and monotone decreasing on the interval $[c, d]$;

(iii) $\mu_{\tilde{N}}(x) = 1$, for each $x \in [b, c]$.

If $\tilde{M}$ is a fuzzy number, then $[\tilde{M}]^{\gamma}$ is a closed interval of $\Re$ for all $\gamma \in [0, 1]$. Here we introduce an alternative notation of $[\tilde{M}]^{\gamma}$ as,

$$[\tilde{M}]^{\gamma} = [m_1(\gamma), m_2(\gamma)] \in \Re .$$

where, $m_1(\gamma)$ and $m_2(\gamma)$ are the *lower* and *upper bounds* of the interval $[\tilde{M}]^{\gamma}$.

A fuzzy set $\tilde{A}$ is called a *triangular fuzzy number* with peak (or center) a, left width $\alpha \ge 0$ and right width $\beta \ge 0$ and is denoted as $\tilde{A} = (a, \alpha, \beta)$, if its membership function has the following form,

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-(a-\alpha)}{\alpha} & \text{if} \quad a-\alpha \le x \le a, \\ \dfrac{(a+\beta)-x}{\beta} & \text{if} \quad a \le x \le a+\beta, \\ 0 & \text{otherwise.} \end{cases}$$

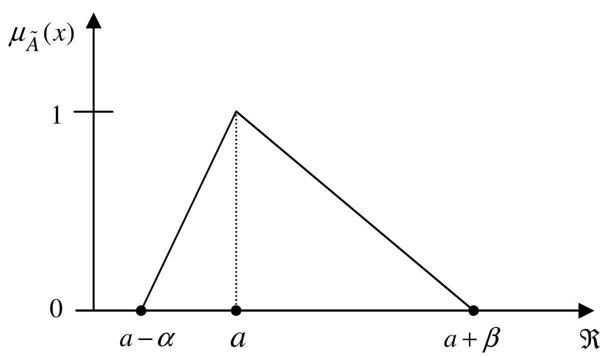

Fig. 1.1: Triangular fuzzy number.

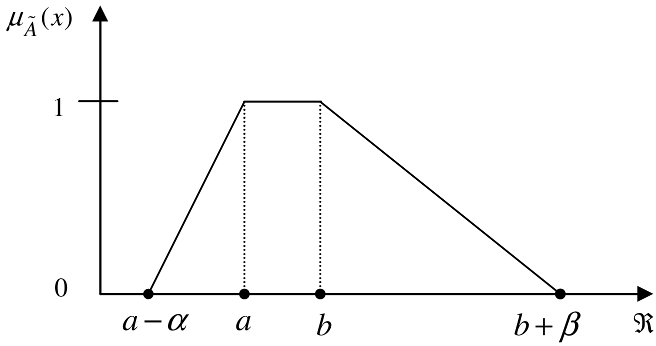

Fig. 1.2: Trapezoidal fuzzy number.

Now, using γ -cut of $\tilde{A}=(a,\ \alpha,\ \beta)$, it can easily be verified that

$$[\tilde{A}]^{\gamma}=[a-(1-\gamma)\alpha,\ a+(1-\gamma)\beta],\quad \forall\gamma\in[0,1].$$

Fig. 1.1 shows a triangular fuzzy number with center *a*, which can be cited as an example of a fuzzy data or fuzzy quantity, '*x* is approximately equal to *a*'.

A fuzzy set $\tilde{A}$ is called a *trapezoidal fuzzy number* with tolerance interval $[a,b]$, $a\le b$, left width $\alpha\ge 0$ and right width $\beta\ge 0$ and is denoted as $\tilde{A}=(a,\ b,\ \alpha,\ \beta)$, if its membership function has the following form

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-(a-\alpha)}{\alpha} & \text{if}\quad a-\alpha\le x\le a,\\ 1 & \text{if}\quad a\le x\le b,\\ \dfrac{(b+\beta)-x}{\beta} & \text{if}\quad a\le x\le b+\beta,\\ 0 & \text{otherwise.}\end{cases}$$

In Fig. 1.2, a trapezoidal fuzzy number is shown which represents the imprecise statement, "*x* is approximately in the interval [*a*, *b*]". A trapezoidal fuzzy number is often called a *flat fuzzy number* or a *fuzzy interval* or *LR-type fuzzy number* (Zadeh (1975), Dubois & Prade (1980)). The trapezoidal fuzzy number reduces to a triangular fuzzy number if $a=b$ and it reduces to an R-fuzzy number, if $\alpha=0$ and an L-fuzzy number, if $\beta=0$.

Similarly, the γ-cut of $\tilde{A}=(a,\ b,\ \alpha,\ \beta)$ can easily be verified that

$$[\tilde{A}]^{\gamma}=[a-(1-\gamma)\alpha,\ \ b+(1-\gamma)\beta],\ \ \forall\gamma\in\ \ [0,\ 1].$$

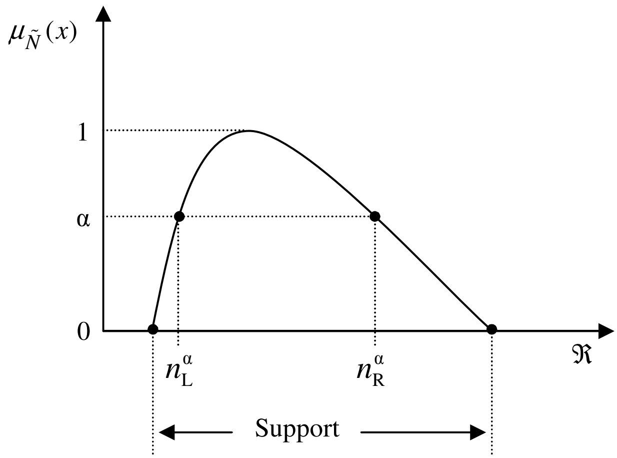

Fig.1.3: The α-cut of a fuzzy number.

Therefore, for a fuzzy set $\tilde{N}$ in X, the symbol $[\tilde{N}]^\alpha$ represents a non-empty non-fuzzy bounded interval contained in X and can be denoted by

$$[\tilde{N}]^\alpha = [n_L^\alpha,\ n_R^\alpha] = \left\{ t \,\middle|\, n_L^\alpha \le t \le n_R^\alpha \right\},$$

where, n_L^α and n_R^α are respectively the lower and upper bounds of the closed interval (Kaufmann & Gupta (1991), Zimmermann (1991)). The essence of the α-cut is that it limits the domain under consideration to the set of elements with the degree of membership of at least alpha. Thus, while the support (extent of fuzziness) of fuzzy set $\tilde{N}$ is its entire base, its α-cut is from n_L^α to n_R^α. Values outside the interval $[n_L^\alpha,\ n_R^\alpha]$ are considered to have a level of membership too insignificant to be relevant and should be excluded (cut out) from consideration. Synonyms of support are degree of fuzziness or a fuzzy spread.

Three fundamental operations in classical set theory are union, intersection and complement. Since membership in a fuzzy set is a matter of degree, set operations are generalized accordingly. Union, intersection and complement operations in fuzzy set theory are similar to disjunction (‘*logical or*’), conjunction (‘*logical and*’) and negation from a logical point of view (Zimmermann (1991), Yen & Langari (1999)).

A common fuzzy disjunction operator is the *maximum* operator and hence, *fuzzy union* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ (Fig. 1.4) is mostly defined as

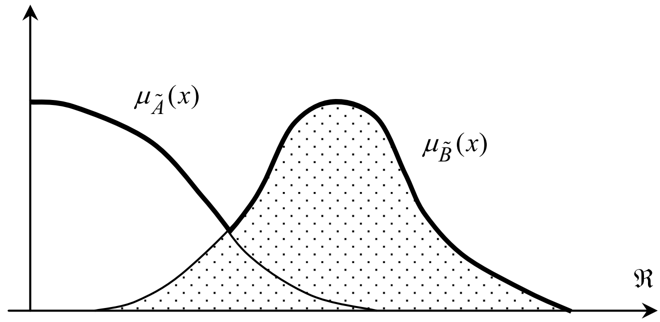

Fig. 1.4: Fuzzy union: the membership function $\mu_{\tilde{A}\cup\tilde{B}}$ is shown by the bold line.

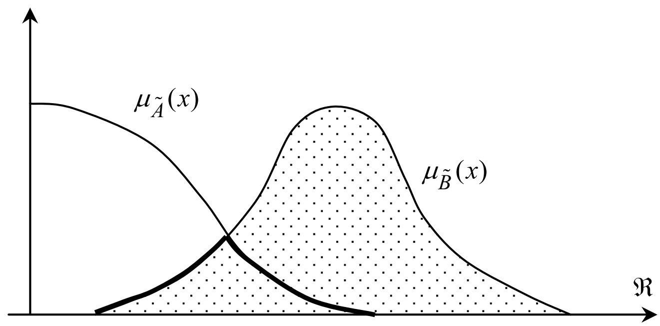

Fig. 1.5: Fuzzy intersection: the membership function $\mu_{\tilde{A}\cap\tilde{B}}$ is shown by the bold line.

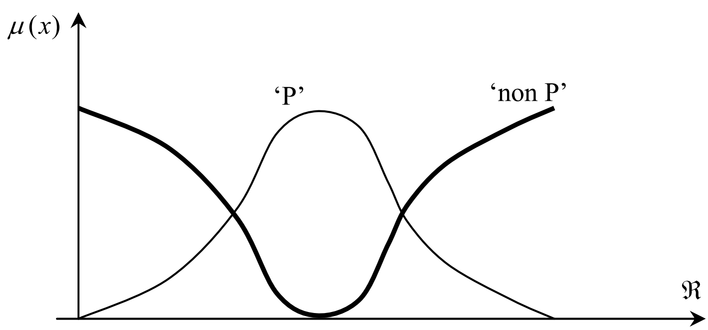

Fig. 1.6: Fuzzy premise 'P' and its complement.