MATHEMATICAL IDEAS, MODELING & APPLICATIONS

VOLUME II OF COMPANION TO CONCRETE MATHEMATICS

Z.A.Melzak

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Volume II of Companion to Concrete Mathematics

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Whatever is worth doing is worth doing badly

GILBERT KEITH CHESTERTON (**1874–1936**)

FOREWORD

Probably every mathematician is acquainted with the embarrassment caused by the simple question: What is mathematics? Representing the most abstract of sciences, we find ourselves unable to give an answer in general, abstract terms. We have to refer to a number of books that give samples of mathematical topics, as does the present book, but it answers a different, although equally puzzling and even more important question: What makes mathematics attractive? The answer has many components, and the book shows them beautifully in their natural interaction. I shall try to separate them by mentioning a few.

First of all, we are given a new awareness of the sources of mathematics. There is, of course, sight, that is, geometry, once synonymous with mathematics and still at the root of very sophisticated, very "modern" problems. But there are other sources: numerous basic tools like looms and hinges or basic crafts like casting, weaving, and measuring. It is both gratifying and surprising to see how many mathematical ideas are implicit in basic human activities. The surprises turn into near-miracles with the applications. To mention only one of many: Why should one expect the Moebius inversion to be connected with the counting of necklaces and therefore, with the counting of amino acids and comma-free codes?

Next, we see mathematics as a part of what Alfred North Whitehead called "adventures of ideas." At every turn of an argument, there is a new view and a new challenge. Starting with an elementary problem, we arrive at deep and general questions merely by looking at the underlying principle of the original problem. Many seemingly difficult problems can be solved by using ideas that are extremely simple once they are discovered. Above all, the ideas provide coherence.

Chapter 4 contains a highly original approach to computing and computability based, as the author puts it, on the method of computing by counting on one's fingers. It is a systematic chapter, and it illustrates another facet of the attractiveness of mathematics: the satisfaction of erecting a tall building on a narrow base.

This is an extraordinary book. It shows the infinite complexity of mathematics, and it enables us to see mathematics as a well-structured unit, as something like a living being in which every organ has a special function and yet serves the whole. The book provides an important service to all of those who want to pass on mathematics to the next generation, not because it is merely entertaining or intriguing, but because it is part of one of the most gratifying of human experiences in the search for insight and truth.

WILHELM MAGNUS

PREFACE

The general purpose of this, the second volume, remains the same as that of the first one. If anything, it is firmer now since, no matter how one interprets it, labor increases resolution. Neither volume is just a collection of problems or even of techniques, in spite of some reviews and other opinions. But then, perhaps the beads have hidden the string.

The concrete element is, of course, emphasized: linkages, casting of convex and other shapes, elliptic functions and the transmission of several telephone conversations on a single wire, an approach to computing and computability based on counting on one's fingers rather than with symbols, a most brutally involved functional equation which, however, arose from focusing the radiation from a point source to a plane beam, pursuit (as of rabbits by hounds) with an inverse which may apply to trendy salesmanship and government stability, wartime radar failure applied to looking at raindrops and giving rise to transport equations possibly generalizing those of the Lotka–Volterra theory of interspecies competition, some tentative extensions of topology arising out of a neurophysiological context, a set-theoretic approach to usury, and so on.

In addition, there is material on formal (not formalistic), manipulative and intuitive aspects of analysis, geometry, and combinatorics. Finally, there is an appendix treating, hopefully not too seriously, some of the "principles" used in volumes 1 and 2. If the subject matter is often elementary or looks antiquated, this is to compensate for the hypermodern mathematical training which builds the individual mathematical house solidly from the roof down, often stopping half-way. This might be due as much to the loss of pleasure in simple things as to the lack of a sense of proportion and reality. Against the second fault gods themselves contend unvictorious, but about the first one something might be done by men. Hence the Chesterton quotation opening this book.

In brief, what is presented here is a subjective distillation of topics found useful or amusing. As the title implies, this book is meant to accompany other texts, books, and instruction or self-instruction. However, several types of courses could conceivably be based on various parts of the combined two volumes:

A course for potential college teachers and instructors with an emphasis on perspective, background, and motivation, illustrating and interconnecting the material taught,

Another one on mathematical techniques and modeling for students of applied mathematics and related sciences or technological disciplines,

A third one, of rather variable nature, as an introduction for promising freshmen and sophomores, encouraging them to start formulating their own problems possibly early, and

A fourth one, for seniors or beginning graduate students, on the linking together of what they may have learned in different courses, including perhaps some supervised writing up of minor research projects on mathematical or historical topics.

One other potential use of the two volumes deserves perhaps a special mention. This concerns a type of retraining for those who have started their mathematical life on a steady pure-abstract diet. Until recently there was, perhaps, no need to change one's course, but we are witnessing a drying up of sources of support for mathematical work as well as an awakening of a sort of mathematical conscience. What should a hypothetical young mathematician do when he wishes to change toward something at least remotely relevant and beneficial in the social sense, and wants to avoid any traps of applied charlatanism as well as those of unbridled purism? The answer is simple and clear, though far from easy: he should develop sufficient breadth to be able to judge for himself. It is my sincere hope that my two books may perhaps help in this respect.

Friends, colleagues, students, well-wishers, correspondents, and others, too numerous to be named one by one, have offered help, suggestions, criticism, and advice. The University of British Columbia granted leave and the Killam Foundation awarded a Fellowship for the school year 1975–1976, during which a heavy part of the work was done. Grateful thanks are hereby tendered to all of them.

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Vancouver, Canada
January 1976

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MATHEMATICAL IDEAS, MODELING AND APPLICATIONS

1
FURTHER TOPICS IN GEOMETRY

1. ELEMENTARY GEOMETRY AND TRIGONOMETRY

(**a**) In this subsection we unite a certain amount of elementary geometrical material: It is shown that some distinguished points of triangles and certain distance-minimizing points can be handled together.

Let similar isosceles triangles with equal angles ω be built outward on the sides of a triangle ABC as shown in Fig. 1. It was mentioned in vol. 1, p. 140 that if $\omega = 60°$ then AA_1, BB_1, CC_1 all meet in P which is then called the Steiner point of the triangle. However, leaving the general ω, let us apply the law of sines to the triangles AKC, AKC_1, ACC_1, and so on; we get then

$$\frac{AK}{KB} = \frac{\sin(A+\omega)\sin B}{\sin(B+\omega)\sin A}$$

and analogous expressions for BL/LC and CM/MA. Multiplying out the three ratios we find that

$$AK \cdot BL \cdot CM = KB \cdot LC \cdot MA.$$

So, by Ceva's theorem (vol. 1, p. 8) the lines AA_1, BB_1, CC_1 are concurrent for *every* ω, not just for $\omega = 60°$. Let the common point of the three segments be $P(\omega)$. This is defined in the first place for $0 < \omega < \pi/2$, but by simple limiting procedures we get also $P(0)$ and $P(\pi/2)$. The points $P(\omega)$ include several important points of the triangle: $P(\pi/3)$ is the Steiner point, $P(\pi/2)$ is the orthocenter (= intersection of the three heights), and $P(0)$ is the center of mass.

The appearance of $P(\pi/3)$ as the Steiner point suggests a connection with an extremum problem: Let $Q = Q(a)$ be that point which minimizes the expression

$$(|QA|^a + |QB|^a + |QC|^a)^{1/a}, \qquad 0 < a < \infty.$$

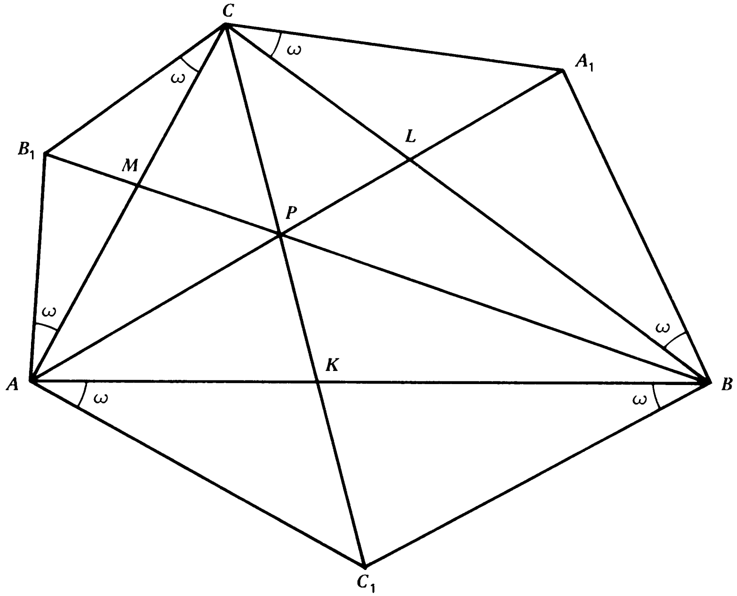

Fig. 1. Isosceles construction.

A limiting procedure shows that $Q(\infty)$ is the largest of the numbers $|QA|$, $|QB|$, $|QC|$. We find now that $Q(1)=P(\pi/3)$, $Q(2)=P(0)$, and $Q(\infty)$ is the circumcenter of the triangle ABC. Several problems suggest themselves; for example, the reader may wish to find the curves described by $P(\omega)$ and by $Q(a)$ as the parameters vary, and to find all their common points. Also, allowing for some modifications, we may extend $P(\omega)$ and $Q(a)$ to negative ω and a. Finally, a generalization to three dimensions might be attempted.

(b) Since by elementary trigonometry

$$\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b$$

we have

$$\sin 3x=4\sin x\sin(60°-x)\sin(60°+x), \tag{1}$$

and replacing x by $30°-x$

$$\cos 3x=4\cos x\cos(60°-x)\cos(60°+x). \tag{2}$$

Placing an equilateral triangle of side 1 with its three vertices on three parallel straight lines as in Fig. 2 we find by (1) and (2)

$$\sin 3x=4ED\cdot CD\cdot CE,\qquad \cos 3x=4AD\cdot AF\cdot DF.$$

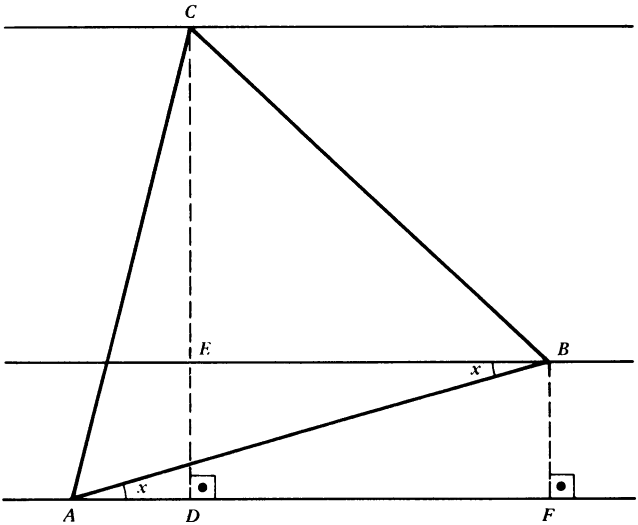

Fig. 2. An equilateral triangle.

Starting with (1) we can produce a simple proof of Morley's theorem: If the angles of a triangle are trisected and the trisectors adjacent to a side are allowed to intersect, then the three intersection points are the vertices of an equilateral triangle (Fig. 3). By taking the angles of the original triangle to be $3x$, $3y$, $3z$ and the circumcircle radius $\frac{1}{2}$, we get the sides to be $\sin 3x$, $\sin 3y$, $\sin 3z$ as shown in the figure. Also, $x+y+z=60°$.

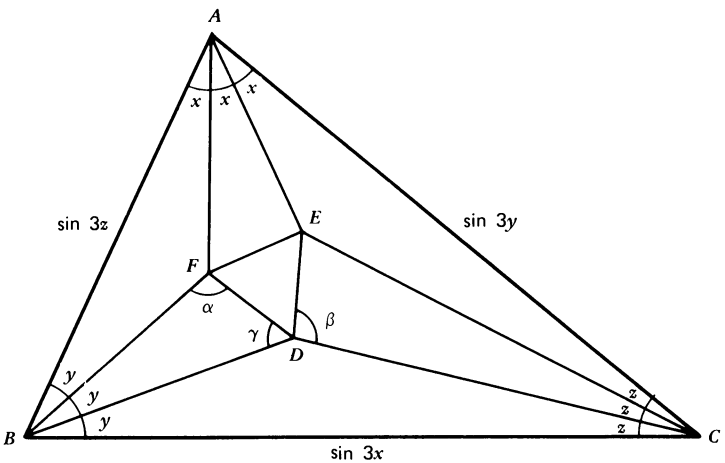

Fig. 3. Morley's theorem.