MEMOIRS
of the
American Mathematical Society

Number 942

Noncommutative Curves of Genus Zero

Related to Finite Dimensional Algebras

Dirk Kussin

September 2009 • Volume 201 • Number 942 (first of 5 numbers) • ISSN 0065-9266

American Mathematical Society

Noncommutative Curves of Genus Zero

Related to Finite Dimensional Algebras

This page intentionally left blank

MEMOIRS
of the
American Mathematical Society

Number 942

Noncommutative Curves of Genus Zero

Related to Finite Dimensional Algebras

Dirk Kussin

September 2009 • Volume 201 • Number 942 (first of 5 numbers) • ISSN 0065-9266

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification*.
Primary 14H45, 16G10; Secondary 14H60, 14A22.

Library of Congress Cataloging-in-Publication Data

Kussin, Dirk, 1967–
Noncommutative curves of genus zero : related to finite dimensional algebras / Dirk Kussin.
p. cm. — (Memoirs of the American Mathematical Society, ISSN 0065-9266 ; no. 942)
"Volume 201, number 942 (first of 5 numbers)."
Includes bibliographical references and index.
ISBN 978-0-8218-4400-7 (alk. paper)
1. Curves, Algebraic. 2. Representations of rings (Algebra) 3. Noncommutative algebras. I. Title.

QA565.K87 2009
516.3′52—dc22 2009019382

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. The 2009 subscription begins with volume 197 and consists of six mailings, each containing one or more numbers. Subscription prices for 2009 are US$709 list, US$567 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of US$65; subscribers in India must pay a postage surcharge of US$95. Expedited delivery to destinations in North America US$57; elsewhere US$160. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the *Notices of the American Mathematical Society*.

Back number information. For back issues see the *AMS Catalog of Publications*.

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2294 USA.

Memoirs of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA.

This publication is indexed in *Science Citation Index*®, *SciSearch*®, *Research Alert*®,
CompuMath Citation Index®, *Current Contents*®/*Physical, Chemical & Earth Sciences*.
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

Dedicated to the memory of my beloved partner
Gordana Stanić

This page intentionally left blank

Contents

Abstract

In these notes we investigate noncommutative smooth projective curves of genus zero, also called exceptional curves. As a main result we show that each such curve $\mathbb{X}$ admits, up to some weighting, a projective coordinate algebra which is a not necessarily commutative *graded factorial domain* R in the sense of Chatters and Jordan. Moreover, there is a natural bijection between the points of $\mathbb{X}$ and the homogeneous prime ideals of height one in R, and these prime ideals are principal in a strong sense.

Curves of genus zero have strong applications in the representation theory of finite dimensional algebras being natural index sets for one-parameter families of indecomposable modules. They play a key role for an understanding of the notion of tameness and conjecturally for an extension of Drozd's Tame and Wild Theorem to arbitrary base fields. The function field of $\mathbb{X}$ agrees with the endomorphism ring of the unique generic module over the associated tame hereditary algebra. This skew field is of finite dimension over its centre which is an algebraic function field in one variable. As another main result we show that the function field is commutative if and only if the multiplicities determined by the homomorphism spaces from line bundles to simples sheaves (originally defined by Ringel for tame hereditary algebras) are equal to one for every point.

The study provides major insights into the nature of arithmetic complications in the representation theory of finite dimensional algebras that arise if the base field is not algebraically closed.

Received by the editor March 19, 2006 and in revised form March 18, 2007.

2000 *Mathematics Subject Classification.* Primary 14H45, 16G10; Secondary 14H60, 14A22, 16S38.

Key words and phrases. noncommutative curve, genus zero, exceptional curve, one-parameter family, separating tubular family, tame bimodule, canonical algebra, tubular algebra, orbit algebra, graded factorial domain, efficient automorphism, ghost automorphism.

This page intentionally left blank

Introduction

Curves of genus zero. In these notes we study noncommutative curves of genus zero. By a curve we always mean a smooth, projective curve defined over a field k. A noncommutative curve is given by a small connected k-category $\mathcal{H}$ which shares the properties with the category $\operatorname{coh}(\mathbb{X})$ of coherent sheaves over a smooth projective curve $\mathbb{X}$, listed below:

- $\mathcal{H}$ is abelian and each object in $\mathcal{H}$ is noetherian.
- All morphism and extension spaces in $\mathcal{H}$ are of finite k-dimension.
- There is an autoequivalence τ on $\mathcal{H}$ (called Auslander-Reiten translation) such that Serre duality $\operatorname{Ext}^1_{\mathcal{H}}(X,Y) = \operatorname{D}\operatorname{Hom}_{\mathcal{H}}(Y,\tau X)$ holds, where $\operatorname{D} = \operatorname{Hom}_k(-,k)$.
- $\mathcal{H}$ contains an object of infinite length.

It follows from Serre duality that $\mathcal{H}$ is a hereditary category, that is, $\operatorname{Ext}^n_{\mathcal{H}}$ vanishes for all $n \geq 2$. Let $\mathcal{H}_0$ be the Serre subcategory of $\mathcal{H}$ formed by the objects of finite length. Then $\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$ (for some index set $\mathbb{X}$) where $\mathcal{U}_x$ are connected uniserial categories, called tubes. The objects in $\mathcal{U}_x$ are called concentrated in x. Of course, any curve should also have the following property.

- $\mathbb{X}$ consists of infinitely many points.

We call $\mathbb{X}$, equipped with $\mathcal{H}$, a noncommutative (smooth, projective) curve.

It follows from the axioms (see [**74**]) that the quotient category $\mathcal{H}/\mathcal{H}_0$ is the category of finite dimensional vector spaces over some skew field $k(\mathcal{H})$, called the function field. We denote it also by $k(\mathbb{X})$. The dimension over $k(\mathcal{H})$ induces the rank of objects in $\mathcal{H}$. The full subcategory of $\mathcal{H}$ of objects which do not contain a subobject of finite length is denoted by $\mathcal{H}_+$; these objects themselves are called (vector) bundles. Bundles of rank one are called line bundles. The category $\mathcal{H}$ has the Krull-Remak-Schmidt property, that is, each object is a finite direct sum of essentially unique indecomposable objects. Moreover, each indecomposable object lies either in $\mathcal{H}_+$ or in $\mathcal{H}_0$.

In the classical case where $\mathbb{X}$ is a smooth projective curve with structure sheaf $\mathcal{O}$, the genus of $\mathbb{X}$ is zero, that is, $\dim_k \operatorname{Ext}^1_{\mathbb{X}}(\mathcal{O},\mathcal{O}) = 0$, if and only if the category $\mathcal{H} = \operatorname{coh}(\mathbb{X})$ contains a tilting object [**69**]. This is an object $T \in \mathcal{H}$ with $\operatorname{Ext}^1_{\mathcal{H}}(T,T) = 0$ and such that $\operatorname{Hom}_{\mathcal{H}}(T,X) = 0 = \operatorname{Ext}^1_{\mathcal{H}}(T,X)$ only holds for $X = 0$.

We therefore say that a noncommutative curve $\mathcal{H}$ is of (absolute) genus zero if

- $\mathcal{H}$ contains a tilting object.

Thus, a noncommutative curve of genus zero is just an exceptional curve as defined in [**68**], a term which we will mainly use in these notes. (These curves are called "exceptional" since the existence of a tilting object is equivalent to the existence of a complete exceptional sequence of objects in $\mathcal{H}$.) In the case of genus zero the request that there are infinitely many points is automatic.

In this setting noncommutativity occurs in two different styles:

(1) The curves are allowed to be "weighted" which gives a parabolic structure on $\mathcal{H}$. This means that there are some points x in which more than one simple object is concentrated. Such a point x is called exceptional; the other points are called homogeneous. We emphasize that for the weighted curves additionally a genus in the orbifold sense (called virtual genus in [**66**]) is of importance.

(2) There is a another kind of noncommutativity of an arithmetic nature, determined by the function field $k(\mathcal{H})$. This skew field is commutative only in very special cases.

The first kind of noncommutativity arising by weights is well-known and the phenomenon is described in its pure form by the weighted projective lines[1] (over an algebraically closed field) defined by Geigle-Lenzing [**34**]. Each weighted curve of genus zero admits only finitely many exceptional points and has an underlying homogeneous curve of genus zero (where all points are homogeneous) from which it arises by so-called insertion of weights. Since this homogeneous curve has the same function field, the homogeneous case and the associated arithmetic effects of noncommutativity are the main topic of these notes.

In the following we assume this homogeneous case, which can be also expressed in the following way.

- For all simple objects $S \in \mathcal{H}$ we have $\operatorname{Ext}^1_{\mathcal{H}}(S,S) \neq 0$ (equivalently, $\tau S \simeq S$).

Such a homogeneous curve $\mathcal{H}$ has genus zero if and only if $\operatorname{Ext}^1_{\mathcal{H}}(L,L) = 0$ for one, equivalently for all line bundles L (which follows from [**74**]). In this case the function field $k(\mathcal{H})$ is of finite dimension over its centre which is an algebraic function field in one variable [**7**]. Moreover, there is a tilting object T which consists of two indecomposable summands, a line bundle L and a further indecomposable bundle $\overline{L}$ so that $\operatorname{Hom}_{\mathcal{H}}(L,\overline{L}) \neq 0$. The endomorphism ring $\operatorname{End}_{\mathcal{H}}(T)$ is a tame hereditary bimodule k-algebra. This underlying bimodule is given as ${}_{\operatorname{End}(\overline{L})}\operatorname{Hom}_{\mathcal{H}}(L,\overline{L})_{\operatorname{End}(L)}$.

We always consider $\mathcal{H}$ together with a fixed line bundle L which we consider as a structure sheaf. This yields a projective coordinate algebra for $\mathcal{H}$, depending on the choice of a suitable endofunctor σ on $\mathcal{H}$, and given as the orbit algebra with respect to L and σ defined as

$$\Pi(L,\sigma) = \bigoplus_{n \geq 0} \operatorname{Hom}_{\mathcal{H}}(L, \sigma^n L),$$

with multiplication given by the rule

$$g * f \overset{def}{=} \sigma^m(g) \circ f,$$

where $f \in \operatorname{Hom}(L, \sigma^m L)$ and $g \in \operatorname{Hom}(L, \sigma^n L)$. Formation of orbit algebras is a standard tool for obtaining projective coordinate algebras in algebraic geometry

[1]Even though in all of these cases we have graded coordinate rings and function fields which are commutative, these curves are nonetheless noncommutative since the coherent sheaves over an affine part correspond to (finitely generated) modules over a ring that is in general not commutative.

(although not under this name) and is frequently used in representation theory, see [**7, 65, 49**]. Note that $\Pi(L,\sigma)$ typically is noncommutative. M. Artin and J. J. Zhang used orbit algebras to define noncommutative projective schemes [**2**] and to prove an analogue of Serre's theorem [**102**].

Let for example σ be the inverse Auslander-Reiten translation τ^-. Then it is easy to see that the pair (L,τ^-) is a so-called ample pair ([**2, 105**]), and thus by the theorem of Artin-Zhang [**2**, Thm. 4.5]

$$\mathcal{H} \simeq \frac{\operatorname{mod}^{\mathbb{Z}}(\Pi(L,\tau^-))}{\operatorname{mod}_0^{\mathbb{Z}}(\Pi(L,\tau^-))},$$

the quotient category modulo the Serre subcategory of $\mathbb{Z}$-graded modules of finite length. Hence $\Pi(L,\tau^-)$ is a projective coordinate algebra for $\mathbb{X}$, and it coincides with the (small) preprojective algebra defined in [**7**]. However the graded algebras constructed in this way are often not practical for studying the geometry of $\mathbb{X}$ explicitly. For example, in the case of the projective line $\mathbb{X}=\mathbb{P}^1(k)$ over k (understood in the scheme sense) we have

$$\Pi(L,\tau^-) = k[X^2, XY, Y^2],$$

which consists of the polynomials in X and Y of even degree. This algebra is a projective coordinate algebras for $\mathbb{P}^1(k)$, as is the full polynomial algebra $k[X,Y]$, graded by total degree. This example illustrates the well-known fact that projective coordinate algebras are not uniquely determined, and also that some projective coordinate algebras are more useful than others. Of the two, only $k[X,Y]$ is graded factorial.

Main results. We show that there exists a graded factorial coordinate algebra in general, given as orbit algebra $\Pi(L,\sigma)$ for a suitable autoequivalence σ on $\mathcal{H}$. Of course, one has to replace the usual factoriality by a noncommutative version.

The geometry of $\mathbb{X}$ is given by the hereditary category $\mathcal{H}$. For this an understanding of the interplay between vector bundles and objects of finite length is important. In particular, with the structure sheaf L, for each point $x\in\mathbb{X}$ and the corresponding simple object $S_x \in \mathcal{U}_x$ the bimodule

$${}_{\operatorname{End}(S_x)}\operatorname{Hom}(L,S_x)_{\operatorname{End}(L)}$$

is of interest. By Serre duality this is equivalent to studying the bimodule

$${}_{\operatorname{End}(L)}\operatorname{Ext}^1(S_x,L)_{\operatorname{End}(S_x)},$$

and this leads directly to the universal extension

$$0 \longrightarrow L \xrightarrow{\pi_x} L(x) \longrightarrow S_x^{e(x)} \longrightarrow 0$$

with the multiplicity (originally defined by Ringel in [**90**])

$$e(x) = [\operatorname{Ext}^1(S_x,L) : \operatorname{End}(S_x)].$$

The above universal extension (for L) is a special case of a more general construction which leads to the tubular shift automorphism σ_x of $\mathcal{H}$, sending an object A to $A(x)$.

We realize the kernels π_x (for each $x\in\mathbb{X}$) as homogeneous elements in a suitable orbit algebra. This is accomplished by an automorphism σ on $\mathcal{H}$ which we call efficient (in 1.1.3). We show that such an automorphism always exists and has the property that for any x the middle term $L(x)$ in the universal extension is of the form $L(x)\simeq\sigma^d(L)$ for some positive integer d, depending on x.

The following theorem provides an explicit one-to-one correspondence between points of $\mathbb{X}$ and homogeneous prime ideals of height one in $\Pi(L,\sigma)$, given by forming universal extensions.

THEOREM. *Let $R = \Pi(L,\sigma)$ with σ being efficient. Let S_x be a simple sheaf concentrated in the point $x \in \mathbb{X}$. Let*

$$0 \longrightarrow L \xrightarrow{\pi_x} \sigma^d(L) \longrightarrow S_x^e \longrightarrow 0$$

be the S_x-universal extension of L. Then the element π_x is normal in R, that is, $R\pi_x = \pi_x R$. Furthermore, $P_x = R\pi_x$ is a homogeneous prime ideal of height one.

Moreover, for any homogeneous prime ideal $P \subset R$ of height one there is a unique point $x \in \mathbb{X}$ such that $P = P_x$.

In this way $\mathbb{X}$ becomes the projective prime spectrum of R. See 1.2.3 and 1.5.1 for the complete statements.

Since a *commutative* noetherian domain is factorial if and only if each prime ideal of height one is principal, we say that a noetherian graded domain R, not necessarily commutative, is a (noncommutative) graded factorial domain if each homogeneous prime ideal of height one is principal, generated by a normal element. This is a graded version of a concept introduced by Chatters and Jordan [**13**].

COROLLARY. *Each homogeneous exceptional curve admits a projective coordinate algebra which is graded factorial.*

The following results clarify the role of the multiplicities $e(x)$. The conclusion is that they measure noncommutativity ("skewness") in several senses:

THEOREM. *The function field of $\mathbb{X}$ is commutative if and only if all multiplicities are equal to one.*

See 4.3.1 for the complete statement; the commutative function fields are explicitly determined. Moreover:

- The multiplicities $e(x)$ are bounded from above by the square root $s(\mathbb{X})$ of the dimension of the function field over its centre. More precisely, if $e^*(x)$ denotes the square root of the dimension of $\mathrm{End}(S_x)$ over its centre, then always $e(x) \cdot e^*(x) \leq s(\mathbb{X})$, and equality holds for all points x except finitely many (2.2.13 and 2.3.5).
- In the graded factorial algebra R we have unique factorization in the sense that each *normal* homogeneous element is an (essentially unique) product of prime elements (which are by definition homogeneous generators of prime ideals of height one). In contrast to the commutative case, a prime element π_x may factorize into a product of several irreducible elements. The number of these factors is essentially given by $e(x)$ (see 1.6.5 and 1.6.6).
- We describe the localization R_P at a prime ideal P. It turns out that R_P is a local ring if and only if the corresponding multiplicity $e(x)$ is one; otherwise R_P is not even semiperfect (2.2.15).

Another surprising phenomenon due to noncommutativity is the occurrence of so-called ghost automorphisms. Denote by $\mathrm{Aut}(\mathbb{X})$ the group of all (isomorphism classes of) automorphisms of the category $\mathcal{H}$ fixing the structure sheaf L. Let $R = \Pi(L,\sigma)$ be the orbit algebra formed with respect to an efficient automorphism σ. Every prime element $\pi_y \in R$ (that is, a normal element generating the prime ideal P_y associated to the point y) induces a graded algebra automorphism γ_y on R, given by the formula $r\pi_y = \pi_y\gamma_y(r)$. This in turn induces an automorphism $\gamma_y^* \in \mathrm{Aut}(\mathbb{X})$ whose action on the set of all points of $\mathbb{X}$ is invisible, but it is a non-trivial element of $\mathrm{Aut}(\mathbb{X})$ if (under an additional assumption, see 3.2.4) for all units u the element $\pi_y u$ is not central. This means that the functor γ_y^* fixes all objects but acts non-trivially on morphisms. Such a functor we call a ghost automorphism.

The simplest example in which this effect arises is given by the curve $\mathbb{X}$ with underlying bimodule $M = {}_{\mathbb{C}}(\mathbb{C} \oplus \overline{\mathbb{C}})_{\mathbb{C}}$ over $k = \mathbb{R}$, where $\mathbb{C}$ acts from the right on the second component via conjugation. A projective coordinate algebra is given by the graded twisted polynomial ring $R = \mathbb{C}[X; Y, \bar{\cdot}]$, graded by total degree, where X is a central variable and for the variable Y we have $Ya = \overline{a}Y$ for all $a \in \mathbb{C}$. We write $R = \mathbb{C}[X, \overline{Y}]$. Then Y is a prime element which is not central (up to units). It follows that complex conjugation induces a ghost automorphism of $\mathbb{X}$. Moreover, denote by σ_x and σ_y the (efficient) tubular shifts corresponding to the points x and y associated with the prime ideals generated by X and Y, respectively. Then $\mathbb{C}[X, \overline{Y}] = \Pi(L, \sigma_x)$ holds.

The following theorem expresses the interrelation between various automorphisms in more detail.

THEOREM. *Let $R = \Pi(L, \sigma)$, where σ is efficient. Let π_y be a prime element of degree d in R, associated to the point y and γ_y the induced graded algebra automorphism. Let σ_y be the tubular shift associated to y. Then there is an isomorphism of functors $\sigma_y \simeq \sigma^d \circ \gamma_y^*$.*

The theorem contains important information about the structure of the Picard group $\mathrm{Pic}(\mathbb{X})$, defined as the subgroup of $\mathrm{Aut}(\mathcal{H})$ generated by all tubular shifts σ_x ($x \in \mathbb{X}$). In particular, in contrast to the algebraically closed case, the Picard group may not be isomorphic to $\mathbb{Z}$.

In Chapter 5 we develop a technique which allows explicit calculation of the automorphism group $\mathrm{Aut}(\mathbb{X})$ in many cases. We illustrate this for the preceding example, where $R = \mathbb{C}[X, \overline{Y}]$. The ghost group is the subgroup of $\mathrm{Aut}(\mathbb{X})$ consisting of all ghost automorphisms.

PROPOSITION. *Let $\mathbb{X}$ be the homogeneous curve with projective coordinate algebra $R = \mathbb{C}[X, \overline{Y}]$. Then $R = \Pi(L, \sigma_x)$, and* $\mathrm{Aut}(\mathbb{X})$ *is generated by*

- *the automorphism γ_y^* of order two, induced by complex conjugation, generating the ghost group;*
- *transformations of the form $Y \mapsto aY$ for $a \in \mathbb{R}_+$;*
- *the automorphism induced by exchanging X and Y.*

Moreover, the Picard group $\mathrm{Pic}(\mathbb{X})$ *is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, and for the Auslander-Reiten translation the following formula holds true:*

$$\tau = \sigma_x^{-1} \circ \sigma_y^{-1} = \sigma_x^{-2} \circ \gamma_y^*.$$

See Sections 5.3 and 5.4 for more general statements. In general the functorial properties of the Auslander-Reiten translation have not been extensively studied. The preceding result shows that interesting effects appear. On objects the Auslander-Reiten translation τ acts like σ_x^{-2}, which agrees with the degree shift by -2. But on morphisms the ghost automorphism induced by complex conjugation enters the game.

So far in this introduction we have concentrated on the homogeneous case. These notes also deal with the weighted case. The following results show that the problem of determining the geometry of an exceptional curve can often be reduced to the homogeneous case.

- We show that insertion of weights into a *central* prime element in a graded factorial coordinate algebra preserves the graded factoriality; the resulting graded algebra is a projective coordinate algebra of a (weighted) exceptional curve (6.2.4).
- The automorphism group of a (weighted) exceptional curve is given by the automorphisms of the underlying homogeneous curve preserving the weights (6.3.1). In particular, both curves have the same ghost group.

The insertion of weights is particularly important for our treatment of the tubular case in Chapter 8. The tubular exceptional curves have a strong relationship to elliptic curves. They are defined by the condition that the so-called virtual (orbifold) genus is one. The main feature of the tubular case is that, very similar to Atiyah's classification of vector bundles over an elliptic curve, $\mathcal{H}$ consists entirely of tubular families. In fact, there is a linear form deg, called the degree, which together with the rank rk defines the slope $\mu(X) = \frac{\deg X}{\operatorname{rk} X}$ of (non-zero) objects X in $\mathcal{H}$. Denote for $q \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by $\mathcal{H}^{(q)}$ the additive closure of indecomposable objects in $\mathcal{H}$ of slope q. Then $\mathcal{H}$ is the additive closure of all $\mathcal{H}^{(q)}$, where $q \in \widehat{\mathbb{Q}}$.

In case the base field is algebraically closed all the tubular families $\mathcal{H}^{(q)}$ are isomorphic to each other as categories, and moreover each is parametrized by the curve $\mathbb{X}$. The reason for this is that in this case the natural action of the automorphism group $\operatorname{Aut}(\mathrm{D}^b(\mathcal{H}))$ on the set $\widehat{\mathbb{Q}}$ is transitive. This is not true in general over an arbitrary base field. We show in Chapter 8 that in general this action may have up to three orbits [**53, 59**]. Accordingly, there are up to three different tubular exceptional curves which are Fourier-Mukai partners.

Another interesting effect treated in the same chapter is the occurrence of line bundles which are not exceptional. Over an algebraically closed field each line bundle L over an exceptional curve $\mathbb{X}$ is exceptional, that is, satisfies $\operatorname{Ext}^1(L, L) = 0$. But this does not extend to arbitrary base fields, the simplest counterexamples existing in the tubular case. We characterize the tubular cases where non-exceptional line bundles exist and show how they can be determined explicitly (Section 8.5).

Applications to finite dimensional algebras. The study of noncommutative curves of genus zero has strong applications in the representation theory of finite dimensional algebras. Conjecturally these curves yield the natural parametrizing sets for one-parameter families of indecomposable modules over finite dimensional tame algebras. This is reflected by the definition of tame algebras over an algebraically closed field k, using as parametrizing curves (affine subsets of) the projective line $\mathbb{P}^1(k)$, and in a certain sense this "explains" that in Drozd's Tame and Wild Theorem [**32, 17**] only rational one-parameter families occur. Note that in the algebraically closed case $\mathbb{P}^1(k)$ is the only homogeneous curve of genus zero.

For the class of tame hereditary algebras and the class of tame canonical algebras [**92**] over an arbitrary field it is well-known that the parametrizing sets are precisely the (affine) curves of genus zero. For a tame algebra, in general more than one exceptional curve is needed to parametrize the indecomposables: there is a tubular (canonical) algebra which requires three such curves (Section 8.3).

It is important to study representation theory over arbitrary base fields since many applications deal with algebras defined over fields which are not algebraically closed. For example, the base field of real numbers is of interest for applications in analysis, the field of rational numbers for number theory, finite fields for the relationships to quantum groups (like Ringel's Hall algebra approach), etc.

When attempting to generalize statements first proven over algebraically closed fields to arbitrary base fields, three typical scenarios of different nature can be observed. Frequently statements and proofs carry over to the more general situation without essential change. Also often the statements remain true but require new proofs, frequently leading to better insights and streamlined arguments even for the algebraically closed case[2]. On the other hand, in a significant number of cases completely new and unexpected effects occur, causing the statements to fail in the general case. The present notes focus in particular on these kinds of new effects.

The representation theoretical analogues of the exceptional curves $\mathbb{X}$ and their hereditary categories $\mathcal{H}$ are given by the concealed canonical algebras [**70**] and their module categories $\operatorname{mod}(\Lambda)$. The link between the two concepts is given by an equivalence $\mathrm{D}^b(\mathcal{H}) \simeq \mathrm{D}^b(\operatorname{mod}(\Lambda))$ of derived categories which leads to a translation between geometric and representation theoretic notions. We illustrate this in the typical case where Λ is a tame hereditary algebra: the subcategory $\mathcal{H}_0$ of objects of finite length corresponds to the full subcategory $\mathcal{R}$ of $\operatorname{mod}(\Lambda)$ formed by the regular representations. Simple objects S_x in $\mathcal{H}$ correspond to simple regular representations. Vector bundles correspond to preprojective (or preinjective) modules, line bundles L to preprojective modules P (or preinjective modules) of defect -1 (or 1, respectively). In particular, the multiplicities $e(x)$ are also definable in terms of preprojective modules of defect -1 and simple regular representations. The function field of $\mathbb{X}$ agrees with the endomorphism ring of the unique generic [**19**] Λ-module. The importance of the generic module for the representation theory of tame hereditary algebras is demonstrated in [**90**]. Our results on exceptional curves all have direct applications to representation theory. In particular:

- Let Λ be a tame hereditary algebra. The (small) preprojective algebra
$$\bigoplus_{n\geq 0} \operatorname{Hom}_\Lambda(P, \tau^{-n}P),$$
where P is a projective module of defect -1 and τ^- is the (inverse) Auslander-Reiten translation on $\operatorname{mod}(\Lambda)$, is a graded factorial domain if the underlying tame bimodule is of dimension type $(1,4)$ (or $(4,1)$)[3]. Note that the (small) preprojective algebra contains the full information on Λ and its representation theory.
- In general there are automorphisms of $\mathrm{D}^b(\operatorname{mod}(\Lambda))$ fixing all objects but acting non-trivially on morphisms, contrary to the algebraically closed case.
- A tubular algebra requires up to three different projective curves of genus zero to parametrize the indecomposable modules.

[2]Some examples for this can be seen in the results of Happel and Reiten about the characterization of hereditary abelian categories with tilting object ([**39**], generalizing [**38**]) and in the proof of the transitivity of the braid group action on complete exceptional sequences for hereditary Artin algebras by Ringel ([**94**], generalizing [**20**]) and by Meltzer and the author for exceptional curves ([**60**], summarized in Section 7.1), generalizing [**78**].

[3]This is also true for many tame bimodules of dimension type $(2,2)$.

- A tubular algebra admits generic modules with up to three different (non-isomorphic) endomorphism rings.
- The endomorphism ring of the generic module over a tame hereditary algebra is commutative if and only if all multiplicities are equal to one, a condition automatically satisfied over algebraically closed base fields. It is surprising that this condition, which essentially says that the morphisms between preprojective and regular representations behave "well", yields the commutativity of the generic module's endomorphism ring, and conversely.

The results on the function field also provide an explanation of the strange fact (pointed out in [**90**]) that a bimodule like ${}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$, given by noncommutative data, leads to a commutative function field

$$\operatorname{Quot}\bigl(\mathbb{R}[U,V]/(U^2+V^2+1)\bigr),$$

whereas a bimodule like ${}_{\mathbb{Q}}\mathbb{Q}(\sqrt{2},\sqrt{3})_{\mathbb{Q}(\sqrt{2},\sqrt{3})}$, given by commutative data, leads to a noncommutative function field, the quotient division ring of

$$\mathbb{Q}\langle U,V\rangle/(UV+VU,\ V^2+2U^2-3).$$

There are a number of inspiring papers dealing with tame hereditary algebras. For example, those by Dlab and Ringel on bimodules and hereditary algebras [**24, 89, 27, 26, 29**] (see additionally [**28, 22, 23**]), in particular Ringel's Rome proceedings paper [**90**], as well as those by Lenzing [**64**], Baer, Geigle and Lenzing [**7**], and by Crawley-Boevey [**18**], dealing with the structure of the parameter curves for tame hereditary algebras over arbitrary fields.

By perpendicular calculus and insertion of weights many problems for concealed canonical algebras (and in particular for tame hereditary algebras) can be reduced to the special class of tame bimodule algebras. This means that we often may restrict our attention to a tame hereditary k-algebra of the form $\Lambda = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix}$, where $M = {}_F M_G$ is a tame bimodule over k, that is, the product of the dimensions of M over the skew fields F and G, respectively, equals 4. These are the analogues of the Kronecker algebra $\begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$, which is isomorphic to the path algebra of the following quiver.

$$\bullet \rightrightarrows \bullet$$

In this homogeneous case $\mathbb{X}$ parametrizes the simple regular representations of Λ. This situation was studied in the cited papers by Dlab and Ringel, by Baer, Geigle and Lenzing, and by Crawley-Boevey. Over the real numbers the structure of $\mathbb{X}$ as topological space is described explicitly in [**24, 25, 26**]. In [**89, 29**] and more generally in [**18**] an affine part of $\mathbb{X}$ is described by the simple modules over a (not necessarily commutative) principal ideal domain. In [**18**] additionally a (commutative) projective curve is constructed, which parametrizes the points of $\mathbb{X}$ and is the centre of the noncommutative projective curves considered in [**64**] and [**7**]. A model-theoretic approach using the Ziegler spectrum is described by Prest [**86**] and Krause [**51**, Chapter 14]. One advantage provided by the present notes is that the geometry of $\mathbb{X}$ is described in terms of graded factorial coordinate algebras. This is useful in particular for studying the properties of the sheaf category $\mathcal{H}$ by forming natural localizations (Chapter 2) and for analyzing the automorphism group of $\mathrm{D}^b(\mathcal{H})$ (Chapter 3). It is also exploited in our proof of the characterization of the commutativity of the function field in terms of the multiplicities (Section 4.3).

We have seen that several new and surprising phenomena occur when an arbitrary base field is allowed. Along the way, we will point out several interesting open problems. The following are particularly worth mentioning:

- Find graded factorial projective coordinate algebras for all weighted cases (by a suitable method of inserting weights also into non-central prime elements).
- Determine the ghost group in general. Describe the action of the Auslander-Reiten translation on morphisms in general.
- The function field $k(\mathbb{X})$ is always of finite dimension over its centre. Is the square root of this dimension always the maximum of the multiplicity function e? Describe each multiplicity $e(x)$ in terms of the function field.
- Is it true that the completions $\widehat{R}$ of the described graded factorial algebras R are factorial again?

These notes are based on the author's Habilitationsschrift with the title "Aspects of hereditary representation theory over non-algebraically closed fields" accepted by the University of Paderborn in 2004. The present version includes further recent results, in particular those concerning the multiplicities in Chapter 2.

We assume that the reader is familiar with the language of representation theory of finite dimensional algebras. We refer to the books of Assem, Simson and Skowroński [**3**], of Auslander, Reiten and Smalø [**5**], and of Ringel [**91**].

Acknowledgements. It is a pleasure to express my gratitude to several people. First I wish to thank Helmut Lenzing for many inspiring discussions on the subject. Most of what I know about representation theory and weighted projective lines I learned from him. He has always encouraged my interest in factoriality questions in this context which started with my doctoral thesis and the generalization [**55**] of a theorem of S. Mori [**83**]. For various helpful discussions and comments I would like to thank Bill Crawley-Boevey, Idun Reiten and Claus Ringel. In particular, some of the results concerning multiplicities were inspired by questions and comments of Bill Crawley-Boevey and Claus Ringel. I also would like to thank Hagen Meltzer for many discussions on several aspects of weighted projective lines and exceptional curves. The section on the transitivity of the braid group action is a short report on a joint work with him. I got the main idea for the definition of an efficient automorphism and thus for the verification of the graded factoriality in full generality during a visit at the Mathematical Institute of the UNAM in Mexico City when preparing a series of talks on the subject. I thank the colleagues of the representation theory group there, in particular Michael Barot and José Antonio de la Peña, for their hospitality and for providing a stimulating working atmosphere. For their useful advices and comments on various parts and versions of the manuscript I thank Axel Boldt, Andrew Hubery and Henning Krause. For her love and her patience I wish to thank my dear partner Gordana.